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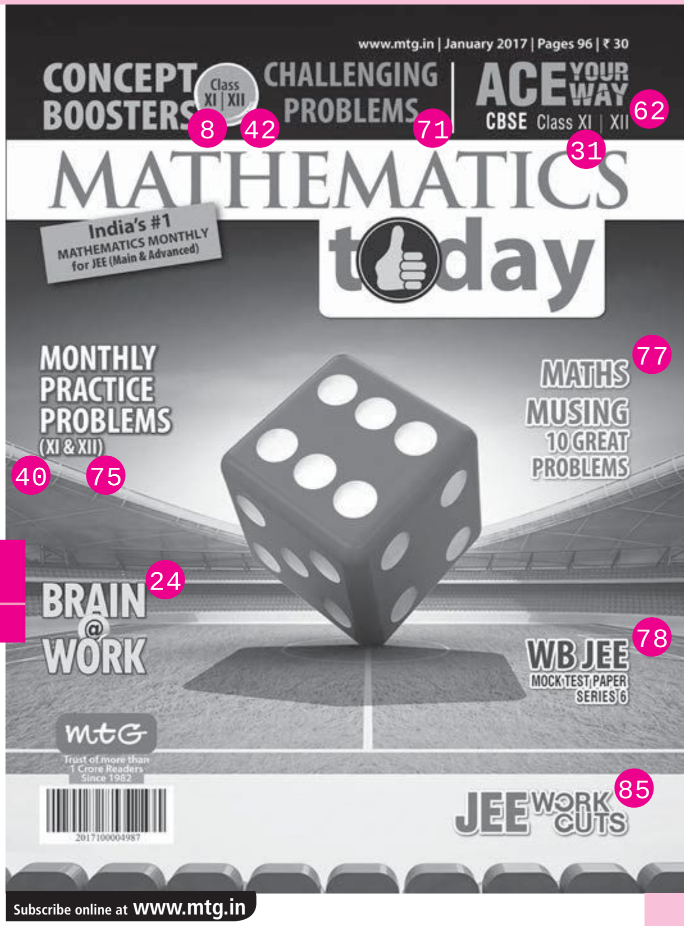

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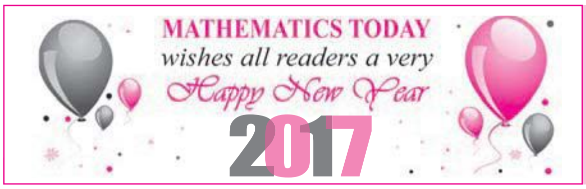

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Editor : Anil Ahlawat

CONCEPT BOOSTERS

Class XI

Circles

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR, B.Tech, IIT Kanpur

A circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane always remains the same *i.e.*, constant.
The fixed point is called the centre of the circle and the fixed distance is called the radius of the circle.

STANDARD FORMS OF EQUATION OF A CIRCLE

- **General equation of a circle :** The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ where g, f, c are constant.
  - Centre of the circle is $(-g, -f)$.
  - Radius of the circle is $\sqrt{g^2 + f^2 - c}$.

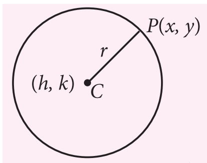
Nature of the circle

- If $g^2 + f^2 - c > 0$, then the radius of the circle will be real. Hence, in this case, it is possible to draw a circle on a plane.
- If $g^2 + f^2 - c = 0$, then the radius of the circle will be zero. Such a circle is known as point circle.
- If $g^2 + f^2 - c < 0$, then the radius of the circle will be an imaginary number. Hence, in this case, it is not possible to draw a circle.

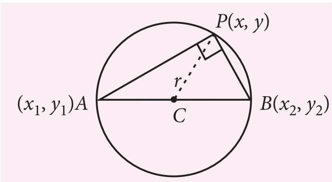
The condition for the second degree equation to represent a circle : The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle iff

- $a = b \neq 0$
- $h = 0$
- $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$
- $g^2 + f^2 - ac \geq 0$

- **Central form of equation of a circle :** The equation of a circle having centre (h, k) and radius r is
  $$(x - h)^2 + (y - k)^2 = r^2$$
  If the centre is origin, then the equation of the circle is $x^2 + y^2 = r^2$.



- **Equation of a circle on a given diameter :** The equation of the circle drawn on the straight line joining two given points (x_1, y_1) and (x_2, y_2) as diameter is
  $$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$



EQUATION OF A CIRCLE IN SOME SPECIAL CASES

- If centre of the circle is (h, k) and it passes through origin then its equation is $(x - h)^2 + (y - k)^2 = h^2 + k^2 \Rightarrow x^2 + y^2 - 2hx - 2ky = 0$.
- If the circle touches x-axis then its equation is $(x \pm h)^2 + (y \pm k)^2 = k^2$.
- If the circle touches y-axis then its equation is $(x \pm h)^2 + (y \pm k)^2 = h^2$.
- If the circle touches both the axes then its equation is $(x \pm r)^2 + (y \pm r)^2 = r^2$.
- If the circle touches x-axis at origin then its equation is $x^2 + (y \pm k)^2 = k^2 \Rightarrow x^2 + y^2 \pm 2ky = 0$.

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- If the circle touches y-axis at origin, the equation of circle is $(x \pm h)^2 + y^2 = h^2 \Rightarrow x^2 + y^2 \pm 2xh = 0$.
- If the circle passes through origin and cuts intercepts a and b on axes, the equation of circle is $x^2 + y^2 - ax - by = 0$ and centre is $C(a/2, b/2)$.

INTERCEPTS ON THE AXES

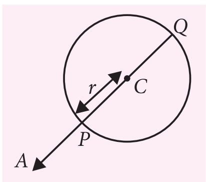
The lengths of intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on x and y axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ respectively. Therefore,

- The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the x-axis in real and distinct points, touches or does not meet in real points according as $g^2 >, =$ or $< c$.
- Similarly, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the y-axis in real and distinct points, touches or does not meet in real points according as $f^2 >, =$ or $< c$.

POSITION OF A POINT WITH RESPECT TO A CIRCLE

A point (x_1, y_1) lies outside, on or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is positive, zero or negative.

The least and greatest distance of a point from a circle: Let $S = 0$ be a circle and $A(x_1, y_1)$ be a point. If the diameter of the circle through A is passing through the circle at P and Q, then



(i) the least distance of A from the circle $= AP = |AC - r|$

(ii) the greatest distance of A from the circle $= AQ = |AC + r|$

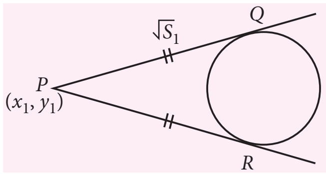
where 'r' is the radius and C is the centre of the circle.

TANGENT TO A CIRCLE AT A GIVEN POINT

- **Point form**
  - The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.
  - The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.
- **Parametric form :** Since parametric co-ordinates of a point on the circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$ then equation of tangent at $(a\cos\theta, a\sin\theta)$ is $x \cdot a\cos\theta + y \cdot a\sin\theta = a^2$ or $x\cos\theta + y\sin\theta = a$.
- **Slope form :** The straight line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$ and the point of contact of tangent $y = mx \pm a\sqrt{1+m^2}$ is $\left(\dfrac{\mp ma}{\sqrt{1+m^2}}, \dfrac{\pm a}{\sqrt{1+m^2}}\right)$.

LENGTH OF TANGENT

Let PQ and PR be two tangents drawn from $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then $PQ = PR$ is called the length of tangent drawn from point P and is given by



$$PQ = PR = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}.$$

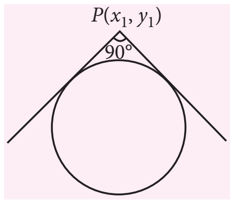
PAIR OF TANGENTS

From a given point $P(x_1, y_1)$ two tangents PQ and PR can be drawn to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$. Their combined equation is $SS_1 = T^2$, where $S = 0$ is the equation of circle, $T = 0$ is the equation of tangent at (x_1, y_1) and S_1 is obtained by replacing x by x_1 and y by y_1 in S.

DIRECTOR CIRCLE

The locus of the point of intersection of two perpendicular tangents to a circle is called the director circle.

Let the circle be $x^2 + y^2 = a^2$, then equation of director circle is $x^2 + y^2 = 2a^2$.



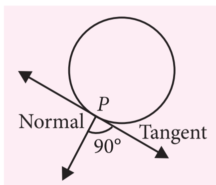
Obviously director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the given circle.

Director circle of $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$.

NORMAL TO A CIRCLE AT A GIVEN POINT

The normal of a circle at any point is a straight line, which is perpendicular to the tangent at the point and always passes through the centre of the circle.

- **Equation of normal:** The equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) is
$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1) \text{ or } \frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}.$$
The equation of normal to the circle $x^2 + y^2 = a^2$ at any point (x_1, y_1) is $xy_1 - x_1 y = 0$ or $\dfrac{x}{x_1} = \dfrac{y}{y_1}$.



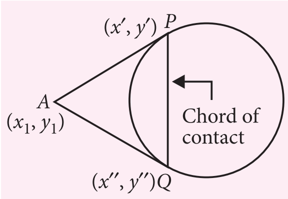
- **Parametric form :** Since parametric co-ordinates of a point on the circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$.

$\therefore$ Equation of normal at $(a\cos\theta, a\sin\theta)$ is
$$\frac{x}{a\cos\theta} = \frac{y}{a\sin\theta} \text{ or } \frac{x}{\cos\theta} = \frac{y}{\sin\theta}$$

or $y = x\tan\theta$ or $y = mx$ where $m = \tan\theta$, which is slope form of normal.

CHORD OF CONTACT OF TANGENTS

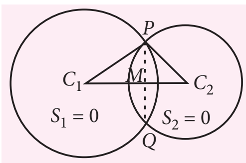
- **Chord of contact :** The chord joining the points of contact of the two tangents to a conic drawn from a given point, outside it, is called the chord of contact of tangents.



- **Equation of chord of contact :** The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.
  Equation of chord of contact at (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.
  It is clear from above that the equation to the chord of contact coincides with the equation of the tangent, if point (x_1, y_1) lies on the circle.
- **Equation of the chord bisected at a given point :** The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at the point (x_1, y_1) is given by $T = S_1$.
  i.e., $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
  $= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$

COMMON CHORD OF TWO CIRCLES

- **Definition :** The chord joining the points of intersection of two given circles is called their common chord.
- **Equation of common chord :** The equation of the common chord of two circles

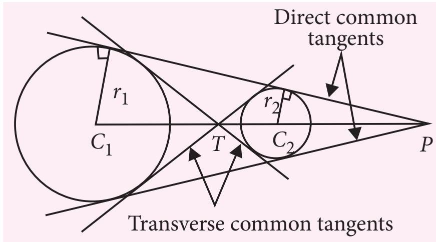


  $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$
  and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$
  is $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$
  i.e., $S_1 - S_2 = 0$
- **Length of the common chord :**
  $PQ = 2(PM) = 2\sqrt{C_1P^2 - C_1M^2}$
  Where C_1P = radius of the circle $S_1 = 0$ and C_1M = length of the perpendicular from the centre C_1 to the common chord PQ.

COMMON TANGENTS TO TWO CIRCLES

Different cases of intersection of two circles :
Let the two circles be $(x - x_1)^2 + (y - y_1)^2 = r_1^2$ .....(i)
and $(x - x_2)^2 + (y - y_2)^2 = r_2^2$ .....(ii)
with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ and radii r_1 and r_2 respectively. Then following cases may arise :

Case I : When $|C_1C_2| > r_1 + r_2$ *i.e.*, the distance between the centres is greater than the sum of radii.



In this case four common tangents can be drawn to the two circles, in which two are direct common tangents and the other two are transverse common tangents.
The points P, T of intersection of direct common tangents and transverse common tangents respectively, always lie on the line joining the centres of the two circles and divide it externally and internally respectively in the ratio of their radii.

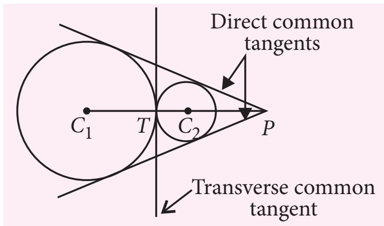
$$\frac{C_1P}{C_2P} = \frac{r_1}{r_2} \text{ (externally) and } \frac{C_1T}{C_2T} = \frac{r_1}{r_2} \text{ (internally)}$$

Hence, the ordinates of P and T are

$$P \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$$

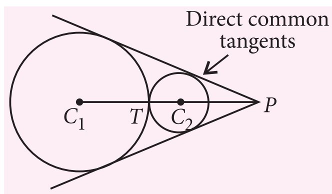
$$\text{and } T \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$

Case II : When $|C_1C_2| = r_1 + r_2$ *i.e.*, the distance between the centres is equal to the sum of radii.

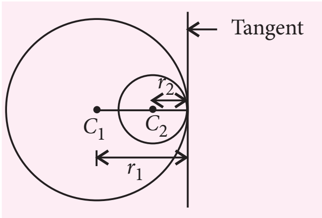


In this case two direct common tangents are real and distinct while the transverse tangents are coincident.

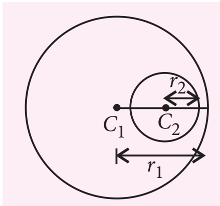
Case III : When $|C_1C_2| < r_1 + r_2$ *i.e.*, the distance between the centres is less than sum of radii. In this case two direct common tangents are real and distinct while the transverse tangents are imaginary.



Case IV : When $|C_1C_2| = |r_1 - r_2|$, *i.e.*, the distance between the centres is equal to the difference of the radii. In this case two tangents are real and coincident while the other two tangents are imaginary.

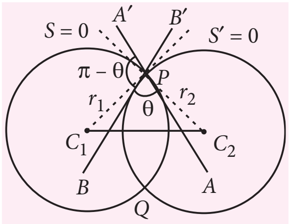


Case V : When $|C_1C_2| < |r_1 - r_2|$, *i.e.*, the distance between the centres is less than the difference of the radii.
In this case, all the four common tangents are imaginary.



ANGLE OF INTERSECTION OF TWO CIRCLES

The angle of intersection between two circles $S = 0$ and $S' = 0$ is defined as the angle between their tangents at their point of intersection.



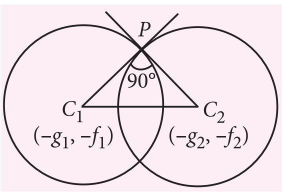
If $\quad S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$\quad S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

are two circles with radii r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by $\cos\theta = \dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$

or $\quad \cos\theta = \dfrac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\,\sqrt{g_2^2 + f_2^2 - c_2}}$.

Condition of Orthogonality :
If the angle of intersection of the two circles is a right angle ($\theta = 90°$), then such circles are called orthogonal circles and condition for orthogonality is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.



FAMILY OF CIRCLES

- The equation of the family of circles passing through the point of intersection of two given circles $S = 0$ and $S' = 0$ is given as $S + \lambda S' = 0$, (where λ is a parameter, $\lambda \neq -1$)
- The equation of the family of circles passing through the point of intersection of circle $S = 0$ and a line $L = 0$ is given as $S + \lambda L = 0$, (where λ is a parameter)
- The equation of the family of circles touching the circle $S = 0$ and the line $L = 0$ at their point of contact P is $S + \lambda L = 0$, (where λ is a parameter)
- The equation of a family of circles passing through two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(where λ is a parameter)

- The equation of family of circles, which touch $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite m is
$(x - x_1)^2 + (y - y_1)^2 + \lambda\{(y - y_1) - m(x - x_1)\} = 0$
And if m is infinite, the family of circles is $(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$, (where λ is a parameter)

RADICAL AXIS

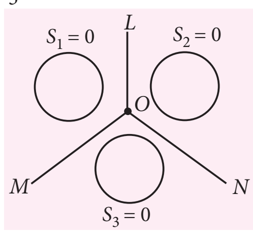
The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.
The equation of the radical axis of the two circles $S_1 = 0$, $S_2 = 0$ is $S_1 - S_2 = 0$ *i.e.*, $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$, which is a straight line.

Properties of the radical axis

- The radical axis and common chord are identical for two intersecting circles.
- The radical axis is perpendicular to the straight line which joins the centres of the circles.
- If two circles cut a third circle orthogonally, the radical axis of the two circles will pass through the centre of the third circle.

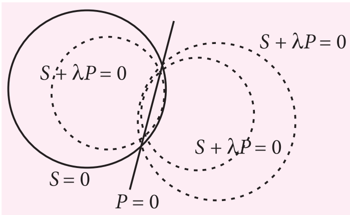
RADICAL CENTRE

The radical axes of three circles, taken in pairs, meet at a point, which is called their radical centre. Let the three circles be
$S_1 = 0$ .....(i), $S_2 = 0$ .....(ii) and $S_3 = 0$ .....(iii)
Let the straight lines *i.e.*, OL and OM meet at O. The equation of any straight line passing through O is $(S_1 - S_2) + \lambda(S_3 - S_1) = 0$, where λ is any constant.



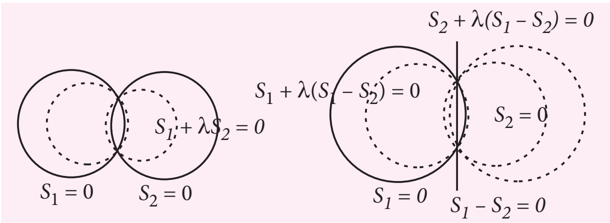
For $\lambda = 1$, this equation become $S_2 - S_3 = 0$, which is, equation of ON.
Thus the third radical axis also passes through the point where the straight lines OL and OM meet.
In the above figure O is the radical centre.

Co-axial system of circles



A system (family) of circles, every pair of which have the same radical axis, are called co-axial circles.

- The equation of a system of co-axial circles, when the equation of the radical axis and of one circle of the system are $P \equiv lx + my + n = 0$, $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ respectively, is $S + \lambda P = 0$ (λ is an arbitrary constant).
- The equation of a co-axial system of circles, where the equation of any two circles of the system are
$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$
and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ respectively, is $S_1 + \lambda(S_1 - S_2) = 0$ or $S_2 + \lambda_1(S_1 - S_2) = 0$
Other form $S_1 + \lambda S_2 = 0$, $(\lambda \neq -1)$



- The equation of a system of co-axial circles in the simplest form is $x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is a constant.

LIMITING POINTS

Limiting points of a system of co-axial circles are the centres of the point circles belonging to the family (Circles whose radii are zero are called point circles).

Let the circle is $x^2 + y^2 + 2gx + c = 0$ .....(i)

where g is a variable and c is a constant.

$\therefore$ Centre and radius of (i) are $(-g, 0)$ and $\sqrt{g^2 - c}$ respectively. Let $\sqrt{g^2 - c} = 0 \Rightarrow g = \pm\sqrt{c}$

Thus we get the two limiting points of the given co-axial system as $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

Clearly the above limiting points are real and distinct, real and coincident or imaginary according as $c >, =, < 0$.

IMPORTANT RESULTS

- If two tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are perpendicular to each other, then $g^2 + f^2 = 2c$.
- If the tangent to the circle $x^2 + y^2 = r^2$ at the point (a, b) meets the coordinate axes at the points A and B and O is the origin, then the area of the triangle OAB is $\dfrac{r^4}{2ab}$.
- The angle between the tangents from (α, β) to the circle $x^2 + y^2 = a^2$ is $2\tan^{-1}\left(\dfrac{a}{\sqrt{\alpha^2 + \beta^2 - a^2}}\right)$.
- If OA and OB are the tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and C is the centre of the circle, then the area of the quadrilateral $OACB$ is $\sqrt{c(g^2 + f^2 - c)}$.
- If the circle $x^2 + y^2 + 2gx + c^2 = 0$ and $x^2 + y^2 + 2fy + c^2 = 0$ touch each other, then $\dfrac{1}{g^2} + \dfrac{1}{f^2} = \dfrac{1}{c^2}$.
- If the line $lx + my + n = 0$ is a tangent to the circle $(x - h)^2 + (y - k)^2 = a^2$, then $(hl + km + n)^2 = a^2(l^2 + m^2)$.
- If O is the origin and OP, OQ are tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the circum-centre of the triangle OPQ is $\left(\dfrac{-g}{2}, \dfrac{-f}{2}\right)$.
- The length of the chord intercept by the circle $x^2 + y^2 = r^2$ on the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ is
$$2\sqrt{\left(\frac{r^2(a^2 + b^2) - a^2b^2}{a^2 + b^2}\right)}$$
- The length of the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = b^2$ is $\dfrac{2ab}{\sqrt{a^2 + b^2}}$.
- The distance between the chord of contact of the tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f) is $\dfrac{1}{2}\dfrac{g^2 + f^2 - c}{\sqrt{g^2 + f^2}}$.
- Locus of mid point of a chords of a circle $x^2 + y^2 = a^2$, which subtends an angle α at the centre is $x^2 + y^2 = (a\cos\alpha/2)^2$.
- The locus of mid point of chords of circle $x^2 + y^2 = a^2$, which are making right angle at centre is $x^2 + y^2 = a^2/2$.

PROBLEMS

Single Correct Answer Type

1. The equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ externally at the point (5, 5), is

(a) $x^2 + y^2 - 18x - 16y - 120 = 0$

(b) $x^2 + y^2 - 18x - 16y + 120 = 0$

(c) $x^2 + y^2 + 18x + 16y - 120 = 0$
(d) $x^2 + y^2 + 18x - 16y + 120 = 0$

2. The equation of the circle which touches x-axis and whose centre is (1, 2), is
(a) $x^2 + y^2 - 2x + 4y + 1 = 0$
(b) $x^2 + y^2 - 2x - 4y + 1 = 0$
(c) $x^2 + y^2 + 2x + 4y + 1 = 0$
(d) $x^2 + y^2 + 4x + 2y + 1 = 0$

3. The locus of the centre of the circle which cuts off intercepts of length $2a$ and $2b$ from x-axis and y-axis respectively, is
(a) $x + y = a + b$ (b) $x^2 + y^2 = a^2 + b^2$
(c) $x^2 - y^2 = a^2 - b^2$ (d) $x^2 + y^2 = a^2 - b^2$

4. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then the radius of the circle is
(a) 3/2 (b) 3/4 (c) 1/10 (d) 1/20

5. If the vertices of a triangle be (2, –2), (–1, –1) and (5, 2), then the equation of its circumcircle is
(a) $x^2 + y^2 + 3x + 3y + 8 = 0$
(b) $x^2 + y^2 - 3x - 3y - 8 = 0$
(c) $x^2 + y^2 - 3x + 3y + 8 = 0$
(d) None of these

6. The equation of a circle which touches both axes and the line $3x - 4y + 8 = 0$ and whose centre lies in the third quadrant is
(a) $x^2 + y^2 - 4x + 4y - 4 = 0$
(b) $x^2 + y^2 - 4x + 4y + 4 = 0$
(c) $x^2 + y^2 + 4x + 4y + 4 = 0$
(d) $x^2 + y^2 - 4x - 4y - 4 = 0$

7. The equation of the circle having centre (1, –2) and passing through the point of intersection of lines $3x + y = 14$ and $2x + 5y = 18$ is
(a) $x^2 + y^2 - 2x + 4y - 20 = 0$
(b) $x^2 + y^2 - 2x - 4y - 20 = 0$
(c) $x^2 + y^2 + 2x - 4y - 20 = 0$
(d) $x^2 + y^2 + 2x + 4y - 20 = 0$

8. Equation of the circle which touches the lines $x = 0$, $y = 0$ and $3x + 4y = 4$ is
(a) $x^2 - 4x + y^2 + 4y + 4 = 0$
(b) $x^2 - 4x + y^2 - 4y + 4 = 0$
(c) $x^2 + 4x + y^2 + 4y + 4 = 0$
(d) $x^2 + 4x + y^2 - 4y + 4 = 0$

9. A circle touches x-axis and cuts off a chord of length $2l$ from y-axis. The locus of the centre of the circle is
(a) a straight line (b) a circle
(c) an ellipse (d) a hyperbola

10. The equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the straight line $y - 4x + 3 = 0$, is
(a) $x^2 + y^2 + 4x - 10y + 25 = 0$
(b) $x^2 + y^2 - 4x - 10y + 25 = 0$
(c) $x^2 + y^2 - 4x - 10y + 16 = 0$
(d) $x^2 + y^2 - 14y + 8 = 0$

11. A circle is concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has area double of its area. The equation of the circle is
(a) $x^2 + y^2 - 6x + 12y - 15 = 0$
(b) $x^2 + y^2 - 6x + 12y + 15 = 0$
(c) $x^2 + y^2 - 6x + 12y + 45 = 0$
(d) None of these

12. The equation of the circle passing through the point (2, 1) and touching y-axis at the origin is
(a) $x^2 + y^2 - 5x = 0$ (b) $2x^2 + 2y^2 - 5x = 0$
(c) $x^2 + y^2 + 5x = 0$ (d) None of these

13. The number of circles touching the lines $x = 0$, $y = a$ and $y = b$ is
(a) One (b) Two (c) Four (d) Infinite

14. The locus of a point which moves such that the sum of the squares of its distances from the three vertices of a triangle is constant, is a circle whose centre is at the
(a) incentre of the triangle
(b) centroid of the triangle
(c) orthocentre of the triangle
(d) None of these

15. The equation of a circle passing through the point (4, 5) and having the centre at (2, 2) is
(a) $x^2 + y^2 + 4x + 4y - 5 = 0$
(b) $x^2 + y^2 - 4x - 4y - 5 = 0$
(c) $x^2 + y^2 - 4x = 13$
(d) $x^2 + y^2 - 4x - 4y + 5 = 0$

16. The locus of the centre of a circle of radius 2 which rolls on the outside of circle $x^2 + y^2 + 3x - 6y - 9 = 0$, is
(a) $x^2 + y^2 + 3x - 6y + 5 = 0$
(b) $x^2 + y^2 + 3x - 6y - 31 = 0$
(c) $x^2 + y^2 + 3x - 6y + \frac{29}{4} = 0$
(d) None of these

17. Area of the circle in which a chord of length $\sqrt{2}$ makes an angle $\frac{\pi}{2}$ at the centre (in sq. units) is

(a) $\frac{\pi}{2}$ (b) 2π (c) π (d) $\frac{\pi}{4}$

18. A line is drawn through a fixed point $P(\alpha, \beta)$ to cut the circle $x^2 + y^2 = r^2$ at A and B. Then $PA \cdot PB$ is equal to

(a) $(\alpha + \beta)^2 - r^2$ (b) $\alpha^2 + \beta^2 - r^2$
(c) $(\alpha - \beta)^2 + r^2$ (d) None of these

19. The equation of the circumcircle of the triangle formed by the lines $x = 0$, $y = 0$, $2x + 3y = 5$ is

(a) $x^2 + y^2 + 2x + 3y - 5 = 0$
(b) $6(x^2 + y^2) - 5(3x + 2y) = 0$
(c) $x^2 + y^2 - 2x - 3y + 5 = 0$
(d) $6(x^2 + y^2) + 5(3x + 2y) = 0$

20. The equation of the circle whose diameter lies on $2x + 3y = 3$ and $16x - y = 4$ which passes through (4,6) is

(a) $5(x^2 + y^2) - 3x - 8y = 200$
(b) $x^2 + y^2 - 4x - 8y = 200$
(c) $5(x^2 + y^2) - 4x = 200$
(d) $x^2 + y^2 = 40$

21. The equation of the circle passing through the point $(-2, 4)$ and through the points of intersection of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ and the line $3x + 2y - 5 = 0$, is

(a) $x^2 + y^2 + 2x - 4y - 4 = 0$
(b) $x^2 + y^2 + 4x - 2y - 4 = 0$
(c) $x^2 + y^2 - 3x - 4y = 0$
(d) $x^2 + y^2 - 4x - 2y = 0$

22. The centre of the circle $x = 2 + 3\cos\theta$, $y = 3\sin\theta - 1$ is

(a) (3, 3) (b) (2, –1) (c) (–2, 1) (d) (–1, 2)

23. Four distinct points $(2k, 3k)$, (1, 0), (0, 1) and (0, 0) lie on a circle for

(a) $\forall k \in I$ (b) $k < 0$
(c) $0 < k < 1$ (d) For two values of k

24. If one end of the diameter is (1, 1) and other end lies on the line $x + y = 3$, then locus of centre of circle is

(a) $x + y = 1$ (b) $2(x - y) = 5$
(c) $2x + 2y = 5$ (d) None of these

25. The line $x\cos\alpha + y\sin\alpha = p$ will be a tangent to the circle $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = 0$, if $p =$

(a) 0 or a (b) $3a$ (c) $2a$ (d) 0 or $2a$

26. The equations of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are

(a) $x = 0, y = 0$
(b) $(h^2 - r^2)x - 2rhy = 0, x = 0$
(c) $y = 0, x = 4$
(d) $(h^2 - r^2)x + 2rhy = 0, x = 0$

27. If the line $lx + my = 1$ be a tangent to the circle $x^2 + y^2 = a^2$, then the locus of the point (l, m) is

(a) a straight line (b) a circle
(c) a parabola (d) an ellipse

28. The equations of the tangents to the circle $x^2 + y^2 = a^2$ parallel to the line $\sqrt{3}x + y + 3 = 0$ are

(a) $\sqrt{3}x + y \pm 2a = 0$ (b) $\sqrt{3}x + y \pm a = 0$
(c) $\sqrt{3}x + y \pm 4a = 0$ (d) None of these

29. The area of the triangle formed by the tangents from the points (h, k) to the circle $x^2 + y^2 = a^2$ and the line joining their points of contact is

(a) $a\frac{(h^2+k^2-a^2)^{3/2}}{h^2+k^2}$ (b) $a\frac{(h^2+k^2-a^2)^{1/2}}{h^2+k^2}$

(c) $\frac{(h^2+k^2-a^2)^{3/2}}{h^2+k^2}$ (d) $\frac{(h^2+k^2-a^2)^{1/2}}{h^2+k^2}$

30. The equations of the normals to the circle $x^2 + y^2 - 8x - 2y + 12 = 0$ at the points whose ordinate is –1, will be

(a) $2x - y - 7 = 0, 2x + y - 9 = 0$
(b) $2x + y + 7 = 0, 2x + y + 9 = 0$
(c) $2x + y - 7 = 0, 2x + y + 9 = 0$
(d) $2x - y + 7 = 0, 2x - y + 9 = 0$

31. Two tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ will be perpendicular to each other, if

(a) $g^2 + f^2 = 2c$ (b) $g = f = c^2$
(c) $g + f = c$ (d) None of these

32. A tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$______ the circle $x^2 + y^2 - 8x + 6y + 20 = 0$

(a) touches (b) cuts at real points
(c) cuts at imaginary points
(d) None of these

33. The line $y = mx + c$ intersects the circle $x^2 + y^2 = r^2$ at two real distinct points, if

(a) $-r\sqrt{1+m^2} < c \le 0$ (b) $0 \le c < r\sqrt{1+m^2}$
(c) both (a) and (b) (d) $-c\sqrt{1-m^2} < r$

Multiple Correct Answer Type

34. If $16l^2 + 9m^2 = 24lm + 6l + 8m + 1$ and S be the equation of circle having $lx + my + 1 = 0$ is tangent then

(a) equation of director circle of S is $x^2 + y^2 - 6x - 8y - 25 = 0$
(b) radius of circle is 5
(c) perpendicular distance from centre of S to $x - y + 1 = 0$ is $\sqrt{2}$
(d) equation of circle S is $x^2 + y^2 + 6x + 8y = 0$

35. A point $P(x, y)$ is called a lattice point if $x, y \in I$ (set of integers). Then the total number of lattice points in the interior of the circle $x^2 + y^2 = a^2$, $a \neq 0$ cannot be

(a) 1996 (b) 1998 (c) 1999 (d) 2001

36. If the line $3x - 4y - \lambda = 0$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$ at (a, b), then $\lambda + a + b$ is equal to

(a) 20 (b) 22 (c) -30 (d) -28

37. In a variable ΔABC, the base BC is fixed and $\angle BAC = \alpha$ (a constant). Then,

(a) the locus of centroid of ΔABC lies on a circle.
(b) the locus of incentre of ΔABC lies on a circle.
(c) the locus of orthocentre of ΔABC lies on a circle.
(d) the locus of excentre opposite to 'A' lies on a circle.

38. A circle is inscribed in a trapezium in which one of the non-parallel sides is perpendicular to the two parallel sides. Then

(a) the diameter of the inscribed circle is the geometric mean of the lengths of the parallel sides.
(b) the diameter of the inscribed circle is the harmonic mean of the lengths of the parallel sides.
(c) the area of the trapezium is the area of the rectangle having lengths of its sides as the lengths of the parallel sides of the trapezium.
(d) the area of the trapezium is half the area of the rectangle having lengths of its sides as the lengths of the parallel sides of the trapezium.

39. Two circles $x^2 + y^2 + px + py - 7 = 0$ and $x^2 + y^2 - 10x + 2py + 1 = 0$ will cut orthogonally if the value of p is

(a) -2 (b) -3 (c) 2 (d) 3

40. The equation of a circle is $S_1 \equiv x^2 + y^2 = 1$. The orthogonal tangents to S_1 meet at another circle S_2 and the orthogonal tangents to S_2 meet at the third circle S_3. Then

(a) Radius of S_2 and S_3 are in the ratio $1:\sqrt{2}$.
(b) Radius of S_2 and S_3 are in the ratio 1 : 2.
(c) The circles S_1, S_2 and S_3 are concentric.
(d) None of these

Comprehension Type

Paragraph for Q. No. 41 to 43

A system of circles is said to be coaxal when every pair of the circles has the same radical axis. It follows from this definition that

1. The centres of all circles of a coaxal system lie on one straight line, which is perpendicular to the common radical axis.

2. Circles passing through two fixed points form a coaxal system with line joining the points as common radical axis.

3. The equation to a coaxal system of which two members are $S_1 = 0$ and $S_2 = 0$ is $S_1 + \lambda S_2 = 0$, λ is parameter. If we choose the line of centres as x-axis and the common radical axis as y-axis, then the simplest form of equation of coaxal circles is

$$x^2 + y^2 + 2gx + c = 0 \quad \text{...(i)}$$

where c is fixed and g is variable.

If $g = \pm\sqrt{c}, c > 0$, then the radius $g^2 - c$ vanishes and the circles become point circles. The points $(\pm\sqrt{c}, 0)$ are called the limiting points of the system of coaxal circles given by (i).

41. The coordinates of the limiting points of the coaxal system to which the circles $x^2 + y^2 + 4x + 2y + 5 = 0$ and $x^2 + y^2 + 2x + 4y + 7 = 0$ belong are

(a) $(0, -3), (0, 3)$ (b) $(0, 3), (-2, -1)$
(c) $(-2, -1), (0, -3)$ (d) $(2, 1), (-2, -1)$

42. The equation to the circle which belongs to the coaxal system of which the limiting points are $(1, -1)$, $(2, 0)$ and which passes through the origin is

(a) $x^2 + y^2 - 4x = 0$ (b) $x^2 + y^2 + 4x = 0$
(c) $x^2 + y^2 - 4y = 0$ (d) $x^2 + y^2 + 4y = 0$

43. If origin be a limiting point of a coaxal system one of whose member is $x^2 + y^2 - 2\alpha x - 2\beta y + c = 0$, then the other limiting point is

(a) $\left(\dfrac{c\alpha}{\alpha^2+\beta^2}, -\dfrac{c\beta}{\alpha^2+\beta^2}\right)$ (b) $\left(\dfrac{c\alpha}{\alpha^2+\beta^2}, \dfrac{c\beta}{\alpha^2+\beta^2}\right)$

(c) $\left(\dfrac{\alpha\beta}{\alpha^2+\beta^2}, \dfrac{c\alpha}{\alpha^2+\beta^2}\right)$ (d) $\left(-\dfrac{c\beta}{\alpha^2+\beta^2}, \dfrac{c\alpha}{\alpha^2+\beta^2}\right)$

Paragraph for Q. No. 44 to 46

Let $ABCD$ is a rectangle with $AB = a$ and $BC = b$. A circle is drawn passing through A and B and touching side CD. Another circle is drawn passing through B and C and touching side AD. Let r_1 and r_2 be the radii of these two circles respectively.

44. r_1 equals

(a) $\dfrac{4b^2 - a^2}{8b}$ (b) $\dfrac{4b^2 + a^2}{8b}$

(c) $\dfrac{4a^2 + b^2}{8a}$ (d) $\dfrac{a^2 - 4b^2}{8a}$

45. $\dfrac{r_1}{r_2}$ equals

(a) $\dfrac{a}{b}\left(\dfrac{4b^2 + a^2}{4a^2 + b^2}\right)$ (b) $\dfrac{b}{a}\left(\dfrac{4a^2 + b^2}{4b^2 + a^2}\right)$

(c) $\frac{a}{b}\left(\frac{4b^2-a^2}{4a^2-b^2}\right)$ (d) $\frac{b}{a}\left(\frac{a^2-4b^2}{4a^2-b^2}\right)$

46. Minimum value of $(r_1 + r_2)$ equals

(a) $\frac{5}{8}(a-b)$ (b) $\frac{5}{8}(a+b)$

(c) $\frac{3}{8}(a-b)$ (d) $\frac{3}{8}(a+b)$

Matrix-Match Type

47. Match the following :

	Column I	Column II	
A.	If a circle passes through $A(1, 0)$ $B(0, -1)$ and $C\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$ such that the tangent at B makes an angle θ with line AB then $\tan\theta$ equals	p.	-4
B.	From a point $(h, 0)$ common tangents are drawn to the circles $x^2 + y^2 = 1$ and the $(x - 2)^2 + y^2 = 4$. The value of h can be	q.	-2
C.	If the common chord of the circle $x^2 + y^2 = 8$ and $(x - a)^2 + y^2 = 8$ subtends right angle at the origin then a can be	r.	1
D.	If the tangents drawn from $(4, k)$ to the circle $x^2 + y^2 = 10$ are at right angles then k can be	s.	2
		t.	4

Integer Answer Type

48. Line segment AC and BD are diameters of circle of radius one. If $\angle BDC = 60°$, the length of line segment AB is

49. The radius of the circles which pass through the point (2, 3) and cut off equal chords of length 6 units along the lines $y - x - 1 = 0$ and $y + x - 5 = 0$ is 'r' then $[r]$ is (where $[\cdot]$ denotes greatest integer function)

50. Let $M(-1, 2)$ and $N(1, 4)$ be two points in a plane rectangular coordinate system XOY. P is a moving point on the x-axis. When $\angle MPN$ takes its maximum value, the x-coordinate of point P is

51. The equation of the tangent at the point $\left(\frac{ab^2}{a^2+b^2}, \frac{a^2b}{a^2+b^2}\right)$ of the circle $x^2+y^2=\frac{a^2b^2}{a^2+b^2}$ is $\frac{x}{a}+\frac{y}{b}=\lambda$, then λ is

52. The line $y = x + c$ will intersect the circle $x^2 + y^2 = 1$ in two coincident points, if then c^2 equals

53. The locus of the point of intersection of the tangents at the extremities of a chord of the circle $x^2+y^2=a^2$ which touches the circle $x^2+y^2=2ax$ is given by $y^2 = k(a - 2x)a$, then, k is

54. The two circles which passes through $(0, a)$ and and $(0, -a)$ touch the line $y = mx + c$ will intersect each other at right angle, then $\frac{c^2-a^2m^2}{a^2}$ equals

55. The equation of the tangent to the circle $x^2 + y^2 = a^2$ which makes a triangle of area a^2 with the coordinate axes, is $x \pm y = ak$, then k^2 is

SOLUTIONS

1. (b) : Let the centre of the required circle be (x_1, y_1) and the centre of given circle is (1, 2). Since radii of both circles are same, therefore, point of contact (5, 5) is the mid point of the line joining the centres of both circles. Hence $x_1 = 9$ and $y_1 = 8$. Hence the required equation is $(x - 9)^2 + (y - 8)^2 = 25$
$\Rightarrow x^2 + y^2 - 18x - 16y + 120 = 0.$

2. (b) : Centre $\equiv (1, 2)$ and since circle touches x-axis, therefore, radius of the circle is 2.
Hence the equation is $(x - 1)^2 + (y - 2)^2 = 2^2$
$\Rightarrow x^2 + y^2 - 2x - 4y + 1 = 0.$

3. (c) : $2\sqrt{g^2-c}=2a$ ...(i)

$2\sqrt{f^2-c}=2b$ ...(ii)

On squaring (i) and (ii) and then subtracting (ii) from (i), we get $g^2 - f^2 = a^2 - b^2$.
Hence the locus is $x^2 - y^2 = a^2 - b^2$.

4. (b) : The diameter of the circle is perpendicular distance between the parallel lines (tangents) $3x - 4y + 4 = 0$ and $3x-4y-\frac{7}{2}=0$ and so it is equal to $\frac{|4+7/2|}{\sqrt{9+16}}=\frac{3}{2}$.

Hence radius is $\frac{3}{4}$.

5. (b) : Let us find the equation of family of circles through $(2, -2)$ and $(-1, -1)$.

i.e., $(x-2)(x+1)+(y+2)(y+1)+\lambda\left(\frac{y+2}{-2+1}-\frac{x-2}{2+1}\right)=0$

Now for point (5, 2) to lie on it,

$3\cdot6+4\cdot3+\lambda\left(\frac{4}{-1}-1\right)=0 \Rightarrow \lambda=\frac{30}{5}=6$

Hence required equation is

$$(x-2)(x+1)+(y+2)(y+1)+6\left(\frac{y+2}{-1}-\frac{x-2}{3}\right)=0$$

or $x^2 + y^2 - 3x - 3y - 8 = 0$.

6. (c) : The equation of circle in third quadrant touching the coordinate axes with centre $(-a, -a)$ and radius 'a' is $x^2 + y^2 + 2ax + 2ay + a^2 = 0$ and we know

$$\left|\frac{3(-a)-4(-a)+8}{\sqrt{9+16}}\right|=a \Rightarrow a=2$$

Hence the required equation is

$x^2 + y^2 + 4x + 4y + 4 = 0$.

7. (a) : The point of intersection of $3x + y - 14 = 0$ and $2x + 5y - 18 = 0$ is $(4, 2)$.

Therefore radius is $\sqrt{9+16}=5$ and equation of the circle is $x^2 + y^2 - 2x + 4y - 20 = 0$.

8. (b) : Let centre of circle be (h, k). Since it touches both axes, therefore $h = k = a$.

Hence equation can be $(x - a)^2 + (y - a)^2 = a^2$

But it also touches the line $3x + 4y = 4$.

Therefore, $\left|\frac{3a+4a-4}{5}\right|=a \Rightarrow a=2$

Hence the required equation of circle is

$x^2 + y^2 - 4x - 4y + 4 = 0$

9. (d) : If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches the x-axis, then $|f|=\sqrt{g^2+f^2-c} \Rightarrow g^2=c$ ...(i)

and cuts a chord of length $2l$ from y-axis

$\Rightarrow 2\sqrt{f^2-c}=2l \Rightarrow f^2-c=l^2$ ...(ii)

Subtracting (i) from (ii), we get $f^2 - g^2 = l^2$.

Hence the locus is $y^2 - x^2 = l^2$, which is obviously a hyperbola.

10. (b) : Let centre of the circle be (h, k), then

$\sqrt{(h-2)^2+(k-3)^2}=\sqrt{(h-4)^2+(k-5)^2}$ ...(i)

and $k - 4h + 3 = 0$ ...(ii)

From (i), we get $h + k - 7 = 0$ ...(iii)

From (ii) and (iii), we get (h, k) as $(2, 5)$. Hence centre is $(2, 5)$ and radius is 2. Now the equation of circle is

$(x - 2)^2 + (y - 5)^2 = 4$

$\Rightarrow \quad x^2 + y^2 - 4x - 10y + 25 = 0$

11. (a) : Equation of circle concentric to given circle is

$x^2 + y^2 - 6x + 12y + k = 0$ ...(i)

Radius of circle (i) $=\sqrt{2}$ (radius of given circle)

$\Rightarrow \quad \sqrt{9+36-k}=\sqrt{2}\sqrt{9+36-15}$

$\Rightarrow \quad 45 - k = 60 \Rightarrow k = -15$

Hence the required equation of circle is

$x^2 + y^2 - 6x + 12y - 15 = 0$

12. (b) : We have the equation of circle

$x^2 + y^2 + 2gx + 2fy + c = 0$

But it passes through $(0, 0)$ and $(2, 1)$, then

$c = 0$ ....(i)

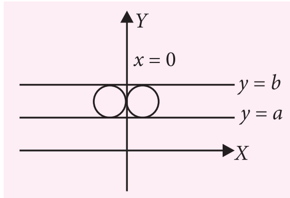
$5 + 4g + 2f = 0$ ....(ii)

Also $\sqrt{g^2+f^2-c}=|g| \Rightarrow \ f=0 \ \{\because c=0\}$

From (ii), $g=-\frac{5}{4}$

Hence the equation will be $2x^2 + 2y^2 - 5x = 0$.

13. (b) : There are only two circles as shown in figure.



14. (b) : Let a triangle has its three vertices as $(0, 0)$, $(a, 0)$, $(0, b)$. We have the moving point (h, k) such that

$h^2 + k^2 + (h - a)^2 + k^2 + h^2 + (k - b)^2 = c$

$\Rightarrow \quad 3h^2 + 3k^2 - 2ah - 2bk + a^2 + b^2 = c$

Therefore, $3x^2 + 3y^2 - 2ax - 2by + a^2 + b^2 = c$

Its centre is $\left(\frac{a}{3},\frac{b}{3}\right)$, which is centroid of triangle.

15. (b) : Given that centre of circle is $(2, 2)$.

Since circle is passing through $(4, 5)$.

So, radius of circle

$$=\sqrt{(4-2)^2+(5-2)^2}=\sqrt{4+9}=\sqrt{13}$$

Therefore, equation of circle is

$$(x-2)^2+(y-2)^2=(\sqrt{13})^2$$

$\Rightarrow \quad x^2 + y^2 - 4x - 4y - 5 = 0$

16. (b) : Let (h, k)be the centre of the circle which rolls on the outside of the given circle. Centre of the given circle is $\left(\frac{-3}{2},3\right)$ and its radius $=\sqrt{\frac{9}{4}+9+9}=\frac{9}{2}$.

Clearly, (h, k) is always at a distance equal to the $\left(\frac{9}{2}+2\right)=\frac{13}{2}$ of the radii of two circles from $\left(-\frac{3}{2},3\right)$.

Therefore $\left(h+\frac{3}{2}\right)^2+(k-3)^2=\left(\frac{13}{2}\right)^2$

$$\Rightarrow \ h^2+k^2+3h-6k+\frac{9}{4}+9-\frac{169}{4}=0$$

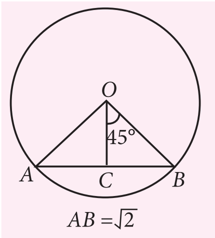
Hence locus of (h, k) is $x^2 + y^2 + 3x - 6y - 31 = 0$.

17. (c) : Let AB be the chord of length $\sqrt{2}$, O be centre of the circle and let OC be the perpendicular from O on AB. Then,

$$AC = BC = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

In ΔOBC, $OB = BC \operatorname{cosec} 45^o$

$$= \frac{1}{\sqrt{2}}.\sqrt{2} = 1$$



Area of the circle $= \pi(OB)^2$

$= \pi$ sq. units.

18. (b) : Let the equation of line through the point (α, β) be $\dfrac{x-\alpha}{\cos\theta} = \dfrac{y-\beta}{\sin\theta} = k$ (say) ...(i)

where k is the distance of any point (x, y) on the line from the point $P(\alpha, \beta)$. Let this line meets the circle $x^2 + y^2 = r^2$ at $(\alpha + k\cos\theta, \beta + k\sin\theta)$.

$\therefore \quad (\alpha + k\cos\theta)^2 + (\beta + k\sin\theta)^2 = r^2$

or $\quad k^2 + 2(\alpha\cos\theta + \beta\sin\theta)k + (\alpha^2 + \beta^2 - r^2) = 0$,

which is a quadratic in k. If k_1 and k_2 are its roots and the line (i) meets circle at A and B, then $PA = k_1$ and $PB = k_2$.

$\therefore \quad PA \cdot PB = k_1 k_2 =$ Products of roots $= \alpha^2 + \beta^2 - r^2$.

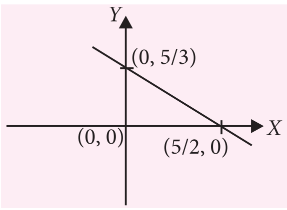
19. (b) : Given,triangle formed by the lines $x = 0$, $y = 0$, $2x + 3y = 5$, so vertices of the triangle are $(0, 0)$, $(5/2, 0)$ and $(0, 5/3)$.

Since circle is passing through $(0, 0)$.

$\therefore$Equation of circle will be

$x^2 + y^2 + 2gx + 2fy = 0$ ...(i)

Also, circle is passing through $(5/2, 0)$ and $(0, 5/3)$



So, $g = -5/4$, $f = -5/6$.

Put the values of g and f in equation (i), we get $6(x^2 + y^2) - 5(3x + 2y) = 0$, which is the required equation of the circle.

20. (a) : Let point (x_1, y_1) on the diameter.

$\Rightarrow \quad 2x_1 + 3y_1 = 3$ ...(i)

and $16x_1 - y_1 = 4$ ...(ii)

On solving (i) and (ii), we get centre,

$$\Rightarrow \quad x_1 = \frac{3}{10}, y_1 = \frac{4}{5}$$

Since, circle passes through $(4, 6)$

$$\text{So, } r^2 = \left(\frac{37}{10}\right)^2 + \left(\frac{26}{5}\right)^2 \Rightarrow r^2 = \frac{4073}{100}$$

$\therefore$ Required equation of circle is

$$\left(x - \frac{3}{10}\right)^2 + \left(y - \frac{4}{5}\right)^2 = \frac{4073}{100}$$

$\Rightarrow \quad 5(x^2 + y^2) - 3x - 8y = 200$.

21. (b) : Required equation of the circle,

$(x^2 + y^2 - 2x - 6y + 6) + \lambda(3x + 2y - 5) = 0$

This circle passing through points $(-2, 4)$, therefore

$(4 + 16 + 4 - 24 + 6) + \lambda(-6 + 8 - 5) = 0, \quad \therefore \quad \lambda = 2$

$\therefore \quad (x^2 + y^2 - 2x - 6y + 6) + 2(3x + 2y - 5) = 0$

$\Rightarrow \quad x^2 + y^2 + 4x - 2y - 4 = 0$

22. (b) : $x = 2 + 3\cos\theta$, $y = 3\sin\theta - 1$

$x^2 + y^2 = 4 + 9\cos^2\theta + 12\cos\theta + 9\sin^2\theta + 1 - 6\sin\theta$

$= 14 + 12\cos\theta - 6\sin\theta$

$= 4(2 + 3\cos\theta) - 2(3\sin\theta - 1) + 4$

$\Rightarrow \quad x^2 + y^2 = 4x - 2y + 4$

$\Rightarrow \quad (x^2 - 4x + 4) + (y^2 + 2y + 1) = 9$

$\Rightarrow \quad (x - 2)^2 + (y + 1)^2 = 9, \therefore$ Centre is $(2, -1)$.

23. (d) : General equation of circle is,

$x^2 + y^2 + 2gx + 2fy + c = 0$

It passes through $(0,0)$, $(1, 0)$ and $(0, 1)$; $\therefore \quad c = 0$

Now $2g + 1 = 0 \Rightarrow g = -\dfrac{1}{2}$ and $2f + 1 = 0 \Rightarrow f = \dfrac{-1}{2}$

Hence equation of circle is

$x^2 + y^2 - x - y = 0$

Since, point $(2k, 3k)$ lies on the circle

$\therefore \quad 4k^2 + 9k^2 - 5k = 0$

$\Rightarrow \quad 13k^2 - 5k = 0 \quad \Rightarrow \quad k = 0$ or $k = \dfrac{5}{13}$

24. (c) : The other end is $(t, 3 - t)$

So the equation of the variable circle is

$(x - 1)(x - t) + (y - 1)(y - 3 + t) = 0$

or $\quad x^2 + y^2 - (1 + t)x - (4 - t)y + 3 = 0$

$\therefore \quad$ The centre (α, β) is given by

$$\alpha = \frac{1+t}{2}, \ \beta = \frac{4-t}{2}$$

$\Rightarrow \quad 2\alpha + 2\beta = 5$

Hence, the locus is $2x + 2y = 5$.

25. (d) : $x\cos\alpha + y\sin\alpha - p = 0$ is a tangent, if perpendicular from centre on it is equal to radius of the circle. Here centre is $(a\cos\alpha, a\sin\alpha)$ and radius is a.

$$\therefore \quad \left|\frac{a\cos^2\alpha + a\sin^2\alpha - p}{\sqrt{1}}\right| = a$$

i.e. $|a - p| = a \Rightarrow p = 0$ or $p = 2a$.

26. (b) : The equation of tangents is $SS_1 = T^2$

$\Rightarrow \quad h^2(x^2 + y^2 - 2rx - 2hy + h^2) = (rx + hy - h^2)^2$

$\Rightarrow \quad (h^2 - r^2)x^2 - 2rhxy = 0 \Rightarrow x\{(h^2 - r^2)x - 2rhy\} = 0$

$\Rightarrow \quad x = 0, (h^2 - r^2)x - 2rhy = 0$

27. (b) : If the line $lx + my - 1$ touches the circle $x^2 + y^2 = a^2$, then applying the condition of tangency,

we have $\dfrac{|l \cdot 0 + m \cdot 0 - 1|}{\sqrt{l^2 + m^2}} = a$

On squaring and simplifying, we get the required locus $x^2+y^2=\frac{1}{a^2}$. Hence it is a circle.

28. (a) : Equation of line parallel to the $\sqrt{3}x+y+3=0$ is $\sqrt{3}x+y+k=0$

But it is a tangent to the circle $x^2 + y^2 = a^2$, then

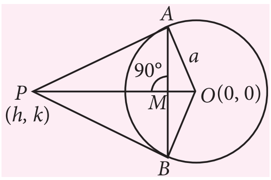
$$\left|\frac{k}{\sqrt{1+3}}\right|=a \Rightarrow k=\pm 2a$$

$\therefore$ Equation of tengent to the circle is

$$\sqrt{3}x+y\pm 2a=0$$

29. (a) : Equation of chord of contact AB is

$xh + yk = a^2$ ...(i)



OM = length of perpendicular from $O(0, 0)$ on line (i)

$$=\frac{a^2}{\sqrt{h^2+k^2}}$$

$$\therefore\ AB=2AM=2\sqrt{OA^2-OM^2}=\frac{2a\sqrt{h^2+k^2-a^2}}{\sqrt{h^2+k^2}}$$

Also PM = length of perpendicular from $P(h, k)$ to the line (i) is $\frac{h^2+k^2-a^2}{\sqrt{h^2+k^2}}$

Therefore, the required area of triangle PAB

$$=\frac{1}{2}\cdot AB\cdot PM=\frac{a(h^2+k^2-a^2)^{3/2}}{h^2+k^2}$$

30. (a) : The abscissa of point is found by substituting the ordinates and solving for abscissa.

$\Rightarrow$ $x^2 - 8x + 15 = 0 \Rightarrow x = 5$ or 3

i.e., Points are (5, –1) and (3, –1).

Normal is given by, $\frac{x-5}{5-4}=\frac{y+1}{-1-1}\Rightarrow 2x+y-9=0$

and $\frac{x-3}{3-4}=\frac{y+1}{-1-1}\Rightarrow 2x-y-7=0$

31. (a) : The equation of tangents will be

$c(x^2 + y^2 + 2gx + 2fy + c) = (gx + fy + c)^2$

These tangents are perpendicular, hence the coefficients of x^2 + coefficients of $y^2 = 0$

$\Rightarrow$ $c - g^2 + c - f^2 = 0$ $\Rightarrow$ $f^2 + g^2 = 2c$.

32. (a) : Tangent is $x - 2y - 5 = 0$ and points of intersection with circle $x^2 + y^2 - 8x + 6y + 20 = 0$ are given by

$4y^2 + 25 + 20y + y^2 - 16y - 40 + 6y + 20 = 0$

$\Rightarrow$ $5y^2 + 10y + 5 = 0$

$\Rightarrow$ $y = -1$ and $x = 3$ *i.e.*, touches.

33. (c) : Substituting equation of line $y = mx + c$ in circle $x^2 + y^2 = r^2$

$\therefore$ $x^2 + (mx + c)^2 = r^2$ $\Rightarrow (1 + m)^2x^2 + 2mxc + c^2 - r^2 = 0$ If discriminant is greater than zero; two real values of x will be obtained.

So, $B^2 > 4AC$

$\Rightarrow$ $4m^2c^2 - 4(c^2 - r^2)(1 + m^2) > 0 \Rightarrow r^2(1 + m^2) > c^2$

$\Rightarrow$ $0\le c<r\sqrt{1+m^2}$ and $-r\sqrt{(1+m^2)}<c\le 0$

34. (a, b) : $16l^2 + 9m^2 = 24lm + 6l + 8m + 1$

$\Rightarrow$ $25(l^2 + m^2) = 9l^2 + 16m^2 + 24lm + 6l + 8m + 1$

$= (3l + 4m + 1)^2$

$$\Rightarrow\ \frac{|l(3)+m(4)+1|}{\sqrt{l^2+m^2}}=5$$

Centre = (3, 4), radius = 5

35. (a, b, c) : Given circle is $x^2 + y^2 = a^2$ ... (i)

Clearly (0, 0) will belong to the interior of circle (i)

Also other points interior to circle (i) will have the co-ordinates of the form $(\pm\alpha, 0)$, $(0, \pm\alpha)$ where $\alpha^2 < a^2$ and $(\pm\alpha, \pm\beta)$, $(\pm\beta, \pm\alpha)$, where $\alpha^2 + \beta^2 < a^2$ and $\alpha, \beta \in I$.

$\therefore$ Number of lattice points in the interior of the circle will be of the form $1 + 4k + 8r$,

Where k, r = 0, 1, 2, ...

Number of such points must be of the form $4m + 1$, where m = 0, 1, 2,......

36. (a, d) : Tangent $\Rightarrow$ $\lambda = 15, -35$

$\lambda = 15$ $\Rightarrow$ $(a, b) = (5, 0)$

$\lambda = -35$ $\Rightarrow$ $(a, b) = (-1, 8)$

37. (a, b, c, d) : $\because$ $\angle A=\alpha$ a constant. If 'I' is incentre of ΔABC.

$\angle BIC=90°+\frac{\alpha}{2}$ which is fixed.

Hence 'I' lies on a fixed circle of which BC is a fixed chord.

$\because$ $\angle A=\alpha$. If 'H' is orthocentre then $\angle BHC = 180° - \alpha$ which is fixed.

Hence, 'H' lies on a circle of which BC is fixed chord.

$\therefore$ $\angle A = \alpha$, If 'G' is centroid of the triangle. $\angle TGK = \alpha$ where T, K are points of trisection of base BC which are fixed.

The fixed line segment TK subtends a constant angle α at a variable point G.

Hence, locus of centroid is also lies on circle.

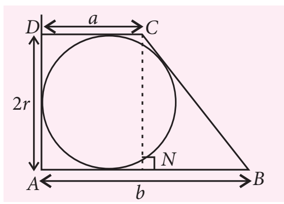
38. (b, c) : $DC + AB = AD + CB$ $\Rightarrow$ $CB = a + b - 2r$

The triangle CNB gives

$(2r)^2 + (b - a)^2 = (a + b - 2r)^2$

Which gives $r=\frac{ab}{a+b}$

$\Rightarrow$ $2r=\frac{2ab}{a+b}$



Area $=\frac{1}{2}(a+b)2r=ab$

39. (c, d) : The given circles will cut orthogonally if

$$2\left(\frac{p}{2}\right)\left(\frac{-10}{2}\right)+2\left(\frac{p}{2}\right)\left(\frac{2p}{2}\right)=-7+1$$

or $p^2-5p+6=0 \Rightarrow p=2$ or 3.

40. (a, c) : Orthogonal tangents to a circle meet at the director circle

$\therefore \quad S_2 \equiv x^2+y^2=2\cdot 1 \Rightarrow S_2 \equiv x^2+y^2=2$

Also, $S_3 \equiv x^2+y^2=4$

Ratio of radius of S_2 and $S_3=\sqrt{2}:2=1:\sqrt{2}$

Also, the three circles are concentric.

41. (c) : The equation of the coaxal system is $x^2+y^2+4x+2y+5+\lambda(x^2+y^2+2x+4y+7)=0$

or $x^2+y^2+\frac{2(2+\lambda)}{1+\lambda}x+\frac{2(1+2\lambda)}{1+\lambda}y+\frac{5+7\lambda}{1+\lambda}=0$

Equating radius to zero, we get

$$\frac{(2+\lambda)^2+(1+2\lambda)^2-(5+7\lambda)(1+\lambda)}{(1+\lambda)^2}=0$$

$\Rightarrow \quad 2\lambda^2+4\lambda=0 \Rightarrow \lambda=0$ or -2

The centre of above system is $\left(-\frac{2+\lambda}{1+\lambda},-\frac{1+2\lambda}{1+\lambda}\right)$

Substituting the values of λ, we get the Coordinates of limiting points $(-2,-1)$ and $(0,-3)$

42. (d) : The point circles represented by the limiting points are $(x-1)^2+(y+1)^2=0$ and $(x-2)^2+y^2=0$

So, the equation of coaxal system is

$(x-1)^2+(y+1)^2+\lambda\{(x-2)^2+y^2\}=0 \quad \text{..... (i)}$

It passes through (0, 0) so, $\lambda=-\frac{1}{2}$

Putting the value of λ in (i) we get the equation to the desired circle as $x^2+y^2+4y=0$.

43. (b) : The equation of the given coaxal system is $x^2+y^2-2\alpha x-2\beta y+c+\lambda(x^2+y^2)=0$

or $x^2+y^2-\frac{2\alpha}{1+\lambda}x-\frac{2\beta}{1+\lambda}y+\frac{c}{1+\lambda}=0$

Its centre is $\left(\frac{\alpha}{1+\lambda},\frac{\beta}{1+\lambda}\right)$ and radius is

$$\frac{\sqrt{\alpha^2+\beta^2-c(1+\lambda)}}{|1+\lambda|}$$

The radius vanishes if $1+\lambda=\frac{\alpha^2+\beta^2}{c}$

So, the other limiting point is $\left(\frac{c\alpha}{\alpha^2+\beta^2},\frac{c\beta}{\alpha^2+\beta^2}\right)$.

(44-46) :

44. (b) **45. (a)** **46. (b)**

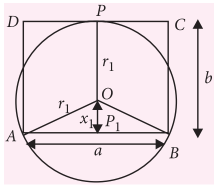
Let $r_1=b-x_1=OP=OA$

$\therefore \quad AP_1=a/2$

$r_1^2=x_1^2+(a/2)^2=(b-x_1)^2$

$\therefore \quad x_1^2+\frac{a^2}{4}=b^2+x_1^2-2bx_1 \Rightarrow x_1=\frac{4b^2-a^2}{8b}$

$\Rightarrow \quad r_1=b-x_1=\frac{4b^2+a^2}{8b}$



Similarly for the circle passing through B and C and touching side AD, $r_2=\frac{4a^2+b^2}{8a}$

Now, $r_1+r_2=\frac{4b^2+a^2}{8b}+\frac{4a^2+b^2}{8a}$

$=\frac{a^3+b^3+4ab(a+b)}{8ab}=\frac{(a+b)(a^2+3ab+b^2)}{8ab}$

$=\frac{(a+b)}{8}\cdot\frac{(a^2-2ab+b^2+5ab)}{ab}=\frac{(a+b)}{8}\cdot\frac{[(a-b)^2+5ab]}{ab}$

But $(a-b)^2 \geq 0$

$\therefore \quad r_1+r_2 \geq \frac{(a+b)}{8}\cdot\frac{5ab}{ab} \Rightarrow r_1+r_2 \geq \frac{5(a+b)}{8}$

47. A $\to$ r; B $\to$ q; C $\to$ p, t; D $\to$ q, s

A. Origin is the circumcentre

$\Rightarrow$ Circle is $x^2+y^2=1 \Rightarrow \theta=\frac{\pi}{4}$

B. A tangent to $x^2+y^2=1$ is $y=mx\pm\sqrt{1+m^2}$. It touches $(x-2)^2+y^2=4$ if $\left|\frac{2m\pm\sqrt{1+m^2}}{\sqrt{1+m^2}}\right|=2$

$\Rightarrow \quad m=\pm\frac{1}{\sqrt{3}}$

$\therefore$ The common tangents are $y=\frac{1}{\sqrt{3}}x+\frac{2}{\sqrt{3}}$ and $y=-\frac{1}{\sqrt{3}}x-\frac{2}{\sqrt{3}}$ which intersect at $(-2, 0)$.

C. Common chord of the given circles is

$(x^2+y^2-8)-[(x-a)^2+y^2-8]=0 \Rightarrow 2x-a=0$

$\Rightarrow \quad \frac{2x}{a}=1$

Since, $x^2+y^2-8=0 \Rightarrow x^2+y^2-8\left(\frac{2x}{a}\right)^2=0$

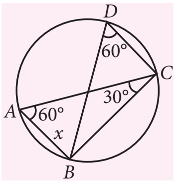
It represents perpendicular lines

$\Rightarrow \quad 1-\frac{32}{a^2}+1=0 \Rightarrow a^2=16 \Rightarrow a=\pm 4$

D. $(4, k)$ must lie on the director circle of the given circle which is $x^2+y^2=20$. Thus

$16+k^2=20 \Rightarrow k=\pm 2$

48. (1) : $\angle A = 60° = \angle D$



$AC = 2$(given), $\angle ABC = 90°$

$\Rightarrow \quad x = 1$

49. (4) : The given two lines pass through the point (2, 3) and are inclined at 45° and 135° to the x–axis. The other ends of chords can easily be calculated as $(2+3\sqrt{2}, 3+3\sqrt{2})$ and $(2-3\sqrt{2}, 3-3\sqrt{2})$

There is symmetry about the line $x = 2$ and therefore the centres of circles lie on $x = 2$.

As the chords subtend right angles at the centre.

$\therefore \quad 2r^2 = 6^2 \Rightarrow \quad r = 3\sqrt{2}$

50. (1) : The centre of a circle passing through points M and N lies on the perpendicular bisector $y = 3 - x$ of MN. Denote the centre by $C(a, 3 - a)$, the equation of the circle is $(x - a)^2 + (y - 3 + a)^2 = 2(1 + a^2)$

Since for a chord with a fixed length the angle at the circumference subtended by the corresponding arc will become larger as the radius of the circle becomes smaller. When $\angle MPN$ reaches its maximum value the circle through the three points M, N and P will be tangent to the x-axis at P, which means

$2(1 + a^2) = (a - 3)^2 \Rightarrow a = 1$ or $a = -7$

Thus the point of contact are $P(1, 0)$ or $P'(-7, 0)$ respectively.

But the radius of circle through the points M, N and P' is larger than that of circle through points M, N and P. Therefore, $\angle MPN > \angle MP'N$. Thus $P = (1, 0)$

$\therefore$ x-coordinate of $P = 1$.

51. (1) : From formula of tangent at a point,

$$x\left(\frac{ab^2}{a^2+b^2}\right)+y\left(\frac{a^2b}{a^2+b^2}\right)=\frac{a^2b^2}{a^2+b^2} \Rightarrow \frac{x}{a}+\frac{y}{b}=1$$

52. (2) : $y = x + c$ intersects at two coincident points, *i.e.*, It is a tangent, therefore $c = \pm\sqrt{2}$.

53. (1) : $T \equiv hx + ky - a^2 = 0$

$$\Rightarrow \quad a = \frac{ah+0-a^2}{\sqrt{h^2+k^2}}$$

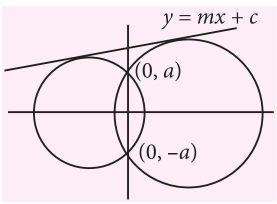
$\Rightarrow \quad h^2 + k^2 = (h - a)^2 \Rightarrow k^2 = a(a - 2h)$

$\therefore \quad$ The locus is $y^2 = a(a - 2x)$.

54. (2) : Equation of circles

$[x^2 + (y - a)(y + a)] + \lambda x = 0$

$\Rightarrow x^2 + y^2 + \lambda x - a^2 = 0$



and $$\sqrt{\left(\frac{\lambda}{2}\right)^2+a^2}=\frac{\frac{-m\lambda}{2}+c}{\sqrt{1+m^2}}$$

$$\Rightarrow (1+m^2)\left[\frac{\lambda^2}{4}+a^2\right]=\left(\frac{m\lambda}{2}-c\right)^2$$

$$\Rightarrow (1+m^2)\left[\frac{\lambda^2}{4}+a^2\right]=\frac{m^2\lambda^2}{4}-mc\lambda+c^2$$

$\Rightarrow \quad \lambda^2 + 4mc\lambda + 4a^2(1 + m^2) - 4c^2 = 0$

$\therefore \quad \lambda_1\lambda_2 = 4[a^2(1 + m^2) - c^2]$

$\Rightarrow \quad g_1g_2 = [a^2(1 + m^2) - c^2]$

and $g_1g_2 + f_1f_2 = \frac{c_1+c_2}{2} \Rightarrow a^2(1+m^2)-c^2=-a^2$

Hence, $\frac{c^2-a^2m^2}{a^2}=2$

55. (2) : Let the tangent be of form $\frac{x}{x_1}+\frac{y}{y_1}=1$ and area of Δ formed by it with coordinate axes is

$\frac{1}{2}x_1y_1 = a^2$ ....(i)

Again, $y_1x + x_1y - x_1y_1 = 0$

Applying conditions of tangency

$$\left|\frac{-x_1y_1}{\sqrt{x_1^2+y_1^2}}\right| = a \text{ or } (x_1^2+y_1^2)=\frac{x_1^2y_1^2}{a^2} \quad ....(ii)$$

From (i) and (ii), we get $x_1 \cdot y_1$; which gives equation of tangent as $x \pm y = \pm a\sqrt{2}$.

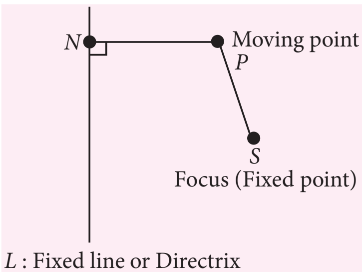
BRAIN @ WORK

PARABOLA

This article is a collection of shortcut methods, important formulas and MCQs along with their detailed solutions which provides an extra edge to the readers who are preparing for various competitive exams like JEE(Main & Advanced) and other PETs.

DEFINITION

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (*i.e.*, focus) is always equal to its distance from a fixed straight line (*i.e.*, directrix) *i.e.* $PS = PN$. The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a parabola if $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 - ab = 0$.



IMPORTANT TERMS

Axis : The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.

Vertex : The points of intersection of the conic section and the axis is (are) called vertex (vertices) of the conic section.

Double Ordinate : A straight line drawn perpendicular to the axis and terminated at both end of the curve is a double ordinate of the conic section.

Latus Rectum : The double ordinate passing through the focus is called the latus rectum of the conic section.

Length of Latus Rectum : Distance between the end points of the latus rectum.

Chord : The line segment joining any two points on the curve.

Focal Chord : Any chord passing through the focus is called focal chord of the conic section.

Normal Chord : A chord that is normal at one end (not neccessarily at other end).

EQUATION OF PARABOLA

(i) If the co-ordinates of the focus is (α, β) and the equation of the directrix is $ax + by + c = 0$ then the equation of the conic section, whose eccentricity $e = 1$ is

$$(x-\alpha)^2 + (y-\beta)^2 = \frac{(ax+by+c)^2}{(a^2+b^2)}$$

$\Rightarrow b^2x^2 + a^2y^2 - 2abxy + x(\text{constant}) + y(\text{constant}) + \text{constant} = 0$

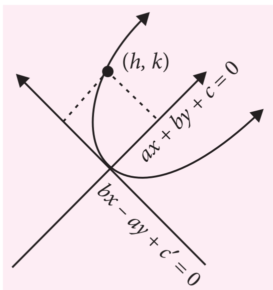
The above equation will reduces to the form $(bx - ay)^2 + 2lx + 2my + n = 0$

The above equation is the general equation of the parabola.

(ii) If the focus is $(a, 0)$ and the equation of directrix is $x + a = 0$ then the equation of parabola is $y^2 = 4ax$.

(iii) If $ax + by + c = 0$ is the equation of the axis and $bx - ay + c' = 0$ is the equation of tangent at the vertex of a parabola and length of latus-rectum is $4a$ then the equation of the parabola is

$$\left(\frac{ax+by+c}{\sqrt{a^2+b^2}}\right)^2 = \pm 4a\left(\frac{bx-ay+c'}{\sqrt{a^2+b^2}}\right)$$



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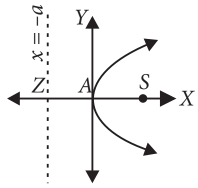
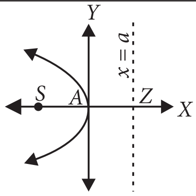
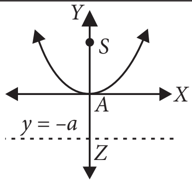
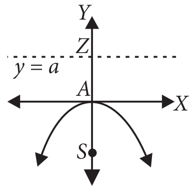
STANDARD EQUATION OF THE PARABOLA

Standard Form	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Graphs				
Eccentricity	$e = 1$	$e = 1$	$e = 1$	$e = 1$
Equation of the Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Equation of Tangents at Vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Coordinates of Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Coordinates of Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of the Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Extremities of Latus Rectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
Length of the Latus Rectum	$4a$	$4a$	$4a$	$4a$
Focal Distance of a Point $(P(x,y))$	$x + a$	$a - x$	$y + a$	$a - y$

PARAMETRIC EQUATION (P.E.) AND PARAMETRIC COORDINATES (P.C.)

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
P.C.	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
P.E.	$x = at^2, \quad y = 2at$	$x = -at^2, y = 2at$	$x = 2at, y = at^2$	$x = 2at, y = -at^2$

EQUATION OF CHORD

Equation of the chord joining the points $(at_1^2, 2at_1)$ and the point $(at_2^2, 2at_2)$ on the curve $y^2 = 4ax$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.

IMPORTANT RESULTS RELATED TO CHORD OF THE PARABOLA

- **Condition of Focal Chord :** $t_1 \cdot t_2 = -1$ where t_1 & t_2 are end points of focal chord.
- **Length of a Focal Chord :**
  (i) $a(t_1 - t_2)^2$ where t_1 & t_2 are end points of focal chord.
  (ii) $a(t + 1/t)^2$ where $(at^2, 2at)$ is one end of the focal chord.
- The length of the smallest focal chord of the parabola is $4a$ which is the length of its latus rectum. Hence, the latus rectum of a parabola is the smallest focal chord.
- The semi latus rectum of the parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord of the parabola.

EQUATION OF THE TANGENT IN DIFFERENT FORMS

(i) Parametric Form

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Eq. of Tangent	$ty = x + at^2$	$ty = -x + at^2$	$tx = y + at^2$	$tx = -y + at^2$
Point of Contact	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Point of Intersection of tangent at t_1 & t_2	$(at_1t_2, a(t_1 + t_2))$	$(-at_1t_2, a(t_1 + t_2))$	$(a(t_1 + t_2), at_1t_2)$	$(a(t_1 + t_2), -at_1t_2)$

(ii) Slope Form

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Eq. of Tangent	$y = mx + a/m$	$y = mx - a/m$	$y = mx - am^2$	$y = mx + am^2$
Point of Contact	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$(2am, am^2)$	$(-2am, -am^2)$
Condition of Tangency	$c = \frac{a}{m}$	$c = -\frac{a}{m}$	$c = -am^2$	$c = am^2$

Some useful Results Related to Tangents

(a) Point 't_1' *i.e.* $(at_1^2, 2at_1)$ point of intersection of tangents at 't_1' & at 't_2' *i.e.* $(at_1t_2, a(t_1+t_2))$ and point 't_2' *i.e.* $(at_2^2, 2at_2)$ taken in order then abscissae of the points are in G.P. and ordinates are in A.P.

(b) Area of the triangle formed by the tangents at t_1 and t_2 and chord joining t_1 and t_2 is $\frac{1}{2}a^2|t_1-t_2|^3$.

(c) The tangents at the extremities of any focal chord of a parabola intersect at right angles at the directrix.

EQUATION OF NORMALS IN DIFFERENT FORMS

(i) Parametric Form

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Eq. of Normal	$y = -tx + 2at + at^3$	$y = tx + 2at + at^3$	$x = -ty + 2at + at^3$	$x = ty + 2at + at^3$
Point of Normality	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Point of Intersection of normals at t_1 & t_2	(m, n)*	$(-m, n)$*	(n, m)*	$(n, -m)$*

* Where, $m = (2a + a(t_1^2 + t_2^2 + t_1t_2))$, & $n = (-at_1t_2(t_1+t_2))$

(ii) Slope form

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Eq. of Normal	$y = mx - 2am - am^3$	$y = mx + 2am + am^3$	$y = mx + 2a + \frac{a}{m^2}$	$y = mx - 2a - \frac{a}{m^2}$
Point of Normality	$(am^2, -2am)$	$(-am^2, 2am)$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$
Condition of Normality	$c = -2am - am^3$	$c = 2am + am^3$	$c = 2a + \frac{a}{m^2}$	$c = -2a - \frac{a}{m^2}$

Some useful Results Related to Normals

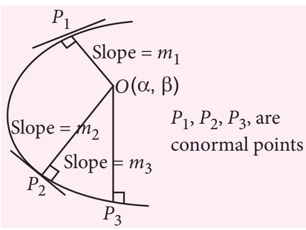
(a) If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ at $(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$

(b) The tangent at one extremity of the focal chord of a parabola is parallel to the normal at the other extremity.

(c) If the normals at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ meet on the parabola, at the point R then $t_1t_2 = 2$.

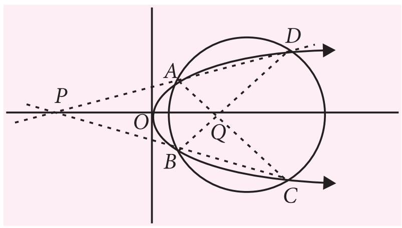
NUMBER OF NORMALS AND CONORMAL POINTS

In general maximum of three real normals can be drawn from any point to the parabola. Let these normals meet the parabola at P_1, P_2 and P_3 respectively. Then the point P_1, P_2 and P_3 are known as conormal points.



RESULTS RELATED TO INTERSECTION OF A CIRCLE & PARABOLA

Depending on the equation of the circle and the parabola these two curve may intersect at maximum of four points. Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the parabola is $y^2 = 4ax$.



Then the following results may hold :

(i) Circle and parabola may intersect maximum at four points.

(ii) The algebraic sum of the ordinates of the four points of intersection of the circle and the parabola is zero.

(iii) The maximum number of common chord of the parabola and the circle is six.

PROBLEMS

1. Tangents drawn to parabola $y^2 = 4ax$ at the points A and B intersect at C. Ordinate of A, C and B forms
(a) a A.P. (b) a G.P.
(c) a H.P. (d) None of these

2. Tangents drawn to parabola $y^2 = 4ax$ at the points A and B intersect at C. If 'S' be the focus of the parabola then, SA, SC and SB forms
(a) a A.P. (b) a G.P.
(c) a H.P. (d) None of these

3. Equation of common tangent of $y^2 = 4ax$ and $x^2 = 4ay$ is
(a) $x + y - a = 0$ (b) $x - y - a = 0$
(c) $x - y + a = 0$ (d) $x + y + a = 0$

4. Maximum number of common chords of a parabola and a circle can be equal to
(a) 2 (b) 4 (c) 6 (d) 8

5. Two perpendicular chords OA and OB of $y^2 = 4ax$ (where 'O' being the origin) are drawn and rectangle $OACB$ is completed. Locus of 'C' is
(a) $y^2 = 4a(x - 4a)$ (b) $y^2 = 4a(x - 8a)$
(c) $y^2 = 2a(x - 4a)$ (d) $y^2 = 2a(x - 8a)$

6. AB is a focal chord of $x^2 - 2x + y - 2 = 0$ whose focus is 'S'. If $AS = l_1$ then BS is equal to
(a) $\dfrac{4l_1}{4l_1 - 1}$ (b) $\dfrac{l_1}{4l_1 - 1}$
(c) $\dfrac{2l_1}{4l_1 - 1}$ (d) None of these

7. If a normal chord of $y^2 = 4ax$ subtends an angle $\pi/2$ at the vertex of the parabola, then its slope is equal to
(a) ± 1 (b) $\pm\sqrt{2}$
(c) ± 2 (d) None of these

8. If a focal chord of $y^2 = 4ax$ makes an angle α, $\alpha \in [0, \pi/4]$ with the positive direction of x-axis, then minimum length of this focal chord (in units) is
(a) $6a$ (b) $2a$
(c) $8a$ (d) None of these

9. If the line $ax + by + c = 0$ is a tangent to the parabola $y^2 - 4y - 8x + 32 = 0$, then $4b^2 =$
(a) $a(7a + 2c + 4b)$ (b) $a(7a + c - 4b)$
(c) $a(7a + 2c - b)$ (d) $a(7a + 2c + b)$

10. The straight line $y = m(x - a)$ will meet the parabola $y^2 = 4ax$ in two distinct real points if
(a) $m \in R$ (b) $m \in [-1, 1]$
(c) $m \in [-\infty, 1] \cup [1, \infty)$ (d) $m \in R - \{0\}$

11. Length of the shortest normal chord of the parabola $y^2 = 4ax$ (in units) is
(a) $a\sqrt{27}$ (b) $3a\sqrt{3}$ (c) $2a\sqrt{27}$ (d) $a\sqrt{3}$

12. $y = 2x + c$, 'c' being variable is a chord of the parabola $y^2 = 4x$, meeting the parabola at A and B. Locus of a point dividing the segment AB internally in the ratio 1 : 1 is
(a) $y = 1$ (b) $x = 1$ (c) $y = 2$ (d) $x = 2$

13. Radius of the circle that passes through origin and touches the parabola $y^2 = 4ax$ at the point $(a, 2a)$ is
(a) $\dfrac{5}{\sqrt{2}}a$ (b) $2\sqrt{2}a$ (c) $5\sqrt{2}a$ (d) $3\sqrt{2}a$

14. The length of latus rectum of the parabola, whose focus is $(a\sin2\theta, a\cos2\theta)$ and directrix is the line $y = a$, is
(a) $|4a\cos^2\theta|$ (b) $|4a\sin^2\theta|$
(c) $|4a\cos^2\theta|$ (d) $|a\cos^2\theta|$

15. Chord AB of the parabola $y^2 = 4ax$ subtends a right angle at the origin. Point of intersection of tangents drawn to parabola at A and B lie on the line
(a) $x + 2a = 0$ (b) $y + 2a = 0$
(c) $x + 4a = 0$ (d) $y + 4a = 0$

16. Mutually perpendicular tangents TA and TB are drawn to $y^2 = 4ax$, minimum length of AB is equal to
(a) $4a$ (b) $6a$ (c) $8a$ (d) $2a$

17. OA and OB are two mutually perpendicular chords of $y^2 = 4ax$, 'O' being the origin. Line AB will always pass through the point
(a) $(2a, 0)$ (b) $(6a, 0)$ (c) $(8a, 0)$ (d) $(4a, 0)$

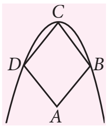
18. If three distinct and real normals can be drawn to $y^2 = 8x$ from the point $(a, 0)$, then
(a) $a > 2$ (b) $a > 4$ (c) $a < 4$ (d) $a \in (2, 4)$

19. PA and PB are tangents drawn to $y^2 = 4ax$ from arbitrary point P. If the angle between tangents is $\pi/4$, then locus of point P is
(a) $y^2 = x^2 + a^2 + 6ax$ (b) $y^2 = x^2 - a^2 + 6ax$
(c) $y^2 = x^2 + a^2 - 6ax$ (d) $y^2 = x^2 - a^2 - 6ax$

20. Angle between the tangents drawn to $y^2 = -4x$, where it is intersected by the line $y = x - 1$ is equal to
(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{6}$ (d) $\dfrac{\pi}{2}$

21. Tangents PA and PB are drawn to $y^2 = -4ax$. If m_{PA} and m_{PB} be the slope of these tangents and $m^2_{PA} + m^2_{PB} = 4$, then locus of 'P' is
(a) $y^2 = 2x(x - a)$ (b) $y^2 = x(x - a)$
(c) $y^2 = 2x(2x - a)$ (d) $y^2 = 2x(x - 2a)$

22. In the adjacent figure a parabola is drawn to pass through the vertices B, C and D of the square $ABCD$. If $A(2, 1)$, $C(2, 3)$, then focus of this parabola is



(a) $\left(1, \frac{11}{4}\right)$ (b) $\left(2, \frac{11}{4}\right)$ (c) $\left(3, \frac{13}{4}\right)$ (d) $\left(2, \frac{13}{4}\right)$

23. Locus of the midpoint of any focal chord of $y^2 = 4ax$ is

(a) $y^2 = a(x - 2a)$ (b) $y^2 = 2a(x - 2a)$
(c) $y^2 = 2a(x - a)$ (d) None of these

24. An equilateral triangle SAB is inscribed in the parabola $y^2 = 4ax$ having its focus at 'S'. If chord AB lies towards the left of S, then side length of this triangle is

(a) $2a(2-\sqrt{3})$ (b) $4a(2-\sqrt{3})$
(c) $2(2-\sqrt{3})$ (d) $8a(2-\sqrt{3})$

25. The parabola $y^2 = 4x$ and the circle $(x - 6)^2 + y^2 = r^2$ will have no common tangent if 'r' is equal to

(a) $r \in (\sqrt{20}, \sqrt{28})$ (b) $r < \sqrt{18}$
(c) $r > \sqrt{18}$ (d) None of these

26. Parabola $y^2 = 4x$ and the circle having its centre at $(6, 5)$ intersect at right angle. Possible point of intersection of these curves can be

(a) $(9, 6)$ (b) $(2, \sqrt{8})$ (c) $(1, 2)$ (d) $(3, 2\sqrt{3})$

27. Tangents and normal drawn to parabola at $A(at^2, 2at)$, $t \neq 0$ meet the x-axis at point B and D respectively. If the rectangle $ABCD$ is completed then locus of 'C' is

(a) $y = 2a$ (b) $y + 2a = 0$
(c) $x = 2a$ (d) None of these

28. Parabolas $y^2 = 4a(x - c_1)$ and $x^2 = 4a(y - c_2)$, where c_1 and c_2 are variable, are such that they touch each other. Locus of their point of contact is

(a) $xy = 2a^2$ (b) $xy = 4a^2$
(c) $xy = a^2$ (d) None of these

29. A variable parabola having length of its latus rectum equal to $4a$ and having its axis parallel to x-axis, is drawn in such a way that it always touches the parabola $y^2 = 4ax$. Locus of the vertex of the variable parabola is

(a) $y^2 = 8ax$ (b) $y^2 + 8ax = 0$
(c) $xy = a^2$ (d) None of these

30. A circle is drawn to pass through the extremities of that latus rectum of the parabola $y^2 = 8x$. It is given that this circle also touches the directrix of the parabola. Thus, radius of this circle is equal to

(a) 4 (b) $\sqrt{21}$ (c) 3 (d) $\sqrt{26}$

31. Locus of the mid point of any normal chords of $y^2 = 4ax$ is

(a) $x = a\left(\frac{4a^2}{y^2} - 2 + \frac{y^2}{2a^2}\right)$ (b) $x = a\left(\frac{4a^2}{y^2} + 2 + \frac{y^2}{2a^2}\right)$

(c) $x = a\left(\frac{4a^2}{y^2} - 2 - \frac{y^2}{2a^2}\right)$ (d) $x = a\left(\frac{4a^2}{y^2} + 2 - \frac{y^2}{2a^2}\right)$

32. The locus of the centre of a circle that passes through $P(a, b)$ and touch the line $y = mx + c$ (it is given that $b \neq ma + c$) is

(a) a straight line (b) circle
(c) parabola (d) hyperbola

33. Slope of the normal chord of $y^2 = 8x$ that get bisected at $(8, 2)$ is

(a) 1 (b) -1 (c) 2 (d) -2

34. Maximum number of common normals of $y^2 = 4ax$ and $x^2 = 4by$ can be equal to

(a) 3 (b) 4 (c) 6 (d) 5

35. If two different tangents of $y^2 = 4x$ are the normals to $x^2 = 4by$, then $|b|$

(a) $> \frac{1}{\sqrt{2}}$ (b) $< \frac{1}{\sqrt{2}}$ (c) $> \frac{1}{2\sqrt{2}}$ (d) $< \frac{1}{2\sqrt{2}}$

SOLUTIONS

1. (a) : If $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, 2at_2)$

$\Rightarrow$ $C \equiv (at_1t_2, a(t_1 + t_2))$

Clearly the ordinates of A, C, B are in A.P.

2. (b) : If $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, 2at_2)$

$\Rightarrow$ $C \equiv (at_1t_2, a(t_1 + t_2))$

Now, $SA = a + at_1^2$, $SB = a + at_2^2$,

$$SC = \sqrt{(at_1t_2 - a)^2 + a^2(t_1 + t_2)^2}$$

$$\Rightarrow SC = a\sqrt{(t_1t_2 - 1)^2 + (t_1 + t_2)^2} = a\sqrt{(1 + t_1^2)(1 + t_2^2)}$$

Clearly, $SC^2 = SA \cdot SB$

3. (d) : Let $y = mx + \frac{a}{m}$ be a tangent of $y^2 = 4ax$.

It will touch $x^2 = 4ay$, provided

$x^2 = 4a\left(mx + \frac{a}{m}\right)$ has equal roots.

$\therefore$ $16a^2m^2 = -16\frac{a^2}{m} \Rightarrow m = -1$

Thus common tangent is $y + x + a = 0$

4. (c) : A circle and a parabola can meet at most in four points. Thus maximum number of common chords is ${}^4C_2$, *i.e.*, 6.

5. (b)

6. (b) : $x^2 - 2x + y - 2 = 0$
$\Rightarrow \quad x^2 - 2x + 1 = 3 - y \Rightarrow (x-1)^2 = -(y-3)$
Length of its latus rectum is 1 unit.

Since AS, $\frac{1}{2}$, BS are in H.P., therefore

$$\frac{1}{2} = \frac{2 \cdot AS \cdot BS}{AS + BS} \Rightarrow BS = \frac{l_1}{(4l_1 - 1)}$$

7. (b) : Let AB be a normal chord, where $A \equiv (at_1^2, 2at_1)$ and $B \equiv (at_2^2, 2at_2)$

We have, $t_2 = -t_1 - \frac{2}{t_1}$ and $t_1t_2 = -4$

$\Rightarrow \quad t_1t_2 = -t_1^2 - 2 = -4 \Rightarrow t_1^2 = 2$

Now, slope of chord $AB = \frac{2}{t_1 + t_2} = -t_1 = \pm\sqrt{2}$

8. (c) : Length of focal chord making an angle 'α' with x-axis is $4a$ cosec$^2\alpha$.

For $a \in \left[0, \frac{\pi}{4}\right]$, its minimum length $= 4a \cdot 2$
$= 8a$ units.

9. (a) : Line will touch the parabola provided

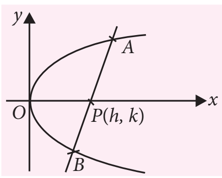
$y^2 - 4y + 32 = \frac{8(-by - c)}{a}$ has equal roots.

$\Rightarrow \quad 4b^2 = 7a^2 + 2a(c + 2b)$

10. (d) : $y = m(x - a)$ passes through the focus $(a, 0)$ of the parabola. Thus for this to be focal chord $m \in R - \{0\}$.

11. (c)

12. (a) : Let $A \equiv (t_1^2, 2t_1)$, $B \equiv (t_2^2, 2t_2)$
Equation of AB is $y(t_1 + t_2) = 2(x + t_1t_2)$
It must be same as $y = 2x + c$



$\Rightarrow \quad \frac{t_1 + t_2}{1} = \frac{2}{2} = \frac{2t_1t_2}{c}$

Let $P(h, k)$ be such that
$AP : PB = 1 : 1$,
then $2h = t_1^2 + t_2^2$, $2k = 2t_1 + 2t_2$
$\Rightarrow \quad t_1 + t_2 = k \quad \therefore \quad k = 1.$
Thus locus of P is $y = 1$.

13. (a) : Equation tangent of parabola at $(a, 2a)$ is
$y - x - a = 0$
Equation of circle touching the parabola at $(a, 2a)$ is
$(x-a)^2 + (y-2a)^2 + \lambda(y - x - a) = 0$
Since, it passes through $(0, 0)$, therefore
$a^2 + 4a^2 + \lambda(-a) = 0 \Rightarrow \lambda = 5a$
Thus, required circle is $x^2 + y^2 - 7ax + ay = 0$

Its radius $= \sqrt{\frac{49}{4}a^2 + \frac{a^2}{4}} = \frac{5}{\sqrt{2}}a$.

14. (b) : Distance of focus from directrix is
$|a \cos 2\theta - a|$
Thus length of latus rectum is $|4a \sin^2\theta|$.

15. (c) : Let $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, 2at_2)$
Thus $t_1t_2 = -4$
Tangents will intersect at $P(at_1t_2, \; a(t_1 + t_2))$.
Thus locus of P is $x = -4a$ or $x + 4a = 0$

16. (a) : Chord of contact of mutually perpendicular tangents is always a focal chord. Thus minimum length of AB is $4a$.

17. (d) : Let $A \equiv (at_1^2, \; 2at_1)$, $B \equiv (at_2^2, \; 2at_2)$.
Thus $t_1t_2 = -4$.
Equation of line AB is $y(t_1 + t_2) = 2(x + at_1t_2)$.
i.e., $y(t_1 + t_2) = 2(x - 4a)$
which clearly passes through a fixed point $(4a, \; 0)$.

18. (b) : Equation of normal in terms of m is
$y = mx - 4m - 2m^3$
If it passes through $(a, 0)$, then
$am - 4m - 2m^3 = 0$

$\Rightarrow \quad m(a - 4 - 2m^2) = 0 \Rightarrow m = 0, \; m^2 = \frac{a-4}{2}$

For three distinct normals,
$a - 4 > 0 \quad \Rightarrow \quad a > 4.$

19. (a)

20. (d) : The line $y = x - 1$ passes through $(1, 0)$ that means it is a focal chord. Hence the required angle is $\pi/2$.

21. (c) : Let coordinates of P be (h, k), then
Equation of tangent of $y^2 = -4ax$, in term of m is

$$x = \frac{y}{m} + \frac{a}{m^2}$$

If it passes through P, then $m^2h - km - a = 0$

$\Rightarrow \quad m_{PA} + m_{PB} = \frac{k}{h}, \; m_{PA} \cdot m_{PB} = -\frac{a}{h}$

Now, $4 = m_{PA}^2 + m_{PB}^2 = (m_{PA} + m_{PB})^2 - 2m_{PA}m_{PB}$

$\Rightarrow \quad 4 = \frac{k^2}{h^2} + \frac{2a}{h}$

Thus, locus is $y^2 = 4x^2 - 2ax$

22. (b) : Clearly, AC is parallel to y-axis. Its midpoint is $(2, 2)$.
Thus $B \equiv (3, 2)$, $D \equiv (1, 2)$
Parabola will be in the form of
$(x - 2)^2 = \lambda(y - 3)$
It passes through $(3, 2)$, therefore $\lambda = -1$
Thus parabola is $(x - 2)^2 = -1(y - 3)$
It focus is $x - 2 = 0, \; y - 3 = -1/4$
i.e., $(2, 11/4)$

23. (c) : Let the midpoint be $P(h, k)$

Equation of this chord is, $T = S_1$

i.e., $yk - 2a(x + h) = k^2 - 4ah$

It must pass through $(a, 0)$

$\Rightarrow \quad -2a(a + h) = k^2 - 4ah$

Thus required locus is, $y^2 = 2ax - 2a^2$

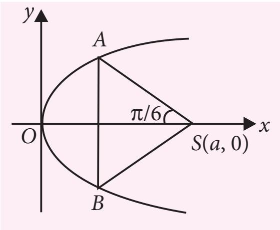
24. (b) : Let $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_1^2, -2at_1)$

We have, $m_{AS} = \tan\left(\dfrac{\pi}{6}\right)$

$\Rightarrow \quad \dfrac{2at_1}{at_1^2 - a} = -\dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad t_1^2 + 2\sqrt{3}t_1 - 1 = 0$

$\Rightarrow \quad t_1 = -\sqrt{3} \pm 2$



Clearly, $t_1 = -\sqrt{3} - 2$ is rejected

Thus, $t_1 = (2 - \sqrt{3})$ $\therefore$ $AB = 4at_1 = 4a(2 - \sqrt{3})$

25. (a) : Any normal of parabola is

$y = -tx + 2t + t^3$

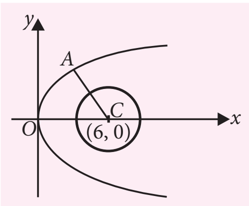
If it pass through (6, 0), then

$-6t + 2t + t^3 = 0$

$\Rightarrow \quad t = 0, t^2 = 4, A \equiv (4, 4)$

Thus for no common tangent

$AC = \sqrt{4+16} < r \Rightarrow r > \sqrt{20}$



26. (a) : Let the possible point be $(t^2, 2t)$

Equation of tangent at this point is

$yt = x + t^2$

It must pass through (6, 5).

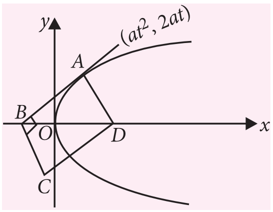
$\Rightarrow \quad t^2 - 5t + 6 = 0 \Rightarrow t = 2, 3$

Thus possible points are (4, 4), (9, 6).

27. (d) : Equations of tangent and normal at A are $yt = x + at^2$ and $y = -tx + 2at + at^3$

$\Rightarrow \quad B \equiv (-at^2, 0)$,

$D \equiv (2a + at^2, 0)$



Since $ABCD$ is a rectangle then midpoints of BD and AC will be coincident.

$\Rightarrow \quad h + at^2 = 2a + at^2 - at^2, k + 2at = 0$

$\Rightarrow \quad h = 2a - at^2, t = -\dfrac{k}{2a} \Rightarrow k^2 + 4ah = 8a^2$

Thus locus is $x^2 + 4ay = 8a^2$.

28. (b) : Let $P(x, y)$ be the point of contact.

At 'P' both of them must have same slope.

We have, $2y\dfrac{dy}{dx} = 4a$ and $2x = 4a\dfrac{dy}{dx}$

Eliminating $\dfrac{dy}{dx}$, we get $xy = 4a^2$

29. (a) : Let the vertex be $P(h, k)$. Then its equation is

$(y - k)^2 = -4a(x - h)$

It will touch $y^2 = 4ax$ provided

$(y - k)^2 = -y^2 + 4ah$ has equal roots.

$\Rightarrow \quad k^2 = 8ah$

Thus locus is $y^2 = 8ax$

30. (a) : Extremities of the latus rectum are (2, 4) and (2, – 4).

Since any circle drawn with any focal chord as its diameter touches the directrix, thus equation of required circle is

$(x - 2)^2 + (y - 4)(y + 4) = 0$

i.e., $x^2 + y^2 - 4x - 12 = 0$

Its radius $= \sqrt{4+12} = 4$

31. (b)

32. (c) : Let $O(h, k)$ be the centre of circle.

Clearly distance of O from P and the line $y = mx + c$ will be equal.

Thus locus of P will be a parabola.

33. (c) : Let AB be the normal chord where $A \equiv (2t_1^2, 4t_1)$, $B \equiv (2t_2^2, 4t_2)$

Its slope $= \dfrac{2}{t_1 + t_2}$

We also have

$t_2 = -t_1 - \dfrac{2}{t_1}$, $16 = 2(t_1^2 + t_2^2)$

and $4 = 4(t_1 + t_2) \Rightarrow t_1 + t_2 = 1$

Thus slope is 2.

34. (d) : Normals to $y^2 = 4ax$ and $x^2 = 4by$ in terms of 'm' are $y = mx - 2am - am^3$

and $y = mx + 2b + \dfrac{b}{m^2}$.

For a common normal

$2b + \dfrac{b}{m^2} + 2am + am^3 = 0$

$\Rightarrow \quad am^5 + 2am^3 + 2bm^2 + b = 0$

That means there can be atmost 5 common normals.

35. (b) : Tangent to $y^2 = 4x$ in terms of m is

$y = mx + \dfrac{1}{m}$

Normal to $x^2 = 4by$ in terms of m is

$y = mx + 2b + \dfrac{b}{m^2}$

If these are same lines, then $\dfrac{1}{m} = 2b + \dfrac{b}{m^2}$

$\Rightarrow \quad 2bm^2 - m + b = 0$

For two different tangents, we must have

$1 - 8b^2 > 0 \Rightarrow |b| < \dfrac{1}{\sqrt{8}}$

CLASS XI Series 9

ACE YOUR WAY CBSE

Statistics | Probability

HIGHLIGHTS

STATISTICS

Measures of Dispersion		Definition/Formulae
Range		It is the difference between two extreme observations of the given data. ∴ Range = Maximum value – Minimum value
Mean Deviation	**For Ungrouped Data**	Let $x_1, x_2, ..., x_n$ be the given n observations. Let $\bar{x}$ be the mean and M be the median. • **About Mean :** $MD(\bar{x}) = \dfrac{\sum_{i=1}^{n} \lvert x_i - \bar{x} \rvert}{n}$ • **About Median :** $MD(M) = \dfrac{\sum_{i=1}^{n} \lvert x_i - M \rvert}{n}$
	For Grouped Data	**For Discrete and Continuous Distribution** • $MD(\bar{x}) = \dfrac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - \bar{x} \rvert$ • $MD(M) = \dfrac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - M \rvert,$ where x_i's are the mid points of the class intervals and $N = \sum_{i=1}^{n} f_i$
	Shortcut Method	• $MD(\bar{x}) = \dfrac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - \bar{x} \rvert$, where $\bar{x} = A + \dfrac{\sum_{i=1}^{n} f_i d_i}{N} \times h,$ $d_i = \dfrac{x_i - A}{h}$, A = assumed mean, h = class size

		• $MD(M)=\frac{1}{N}\sum_{i=1}^{n} f_i\lvert x_i - M\rvert,$ where $M = l + \left(\frac{\frac{N}{2}-c}{f}\right)\times h$ l, f and h are respectively the lower limit, the frequency and the width of the median class and c is the cumulative frequency of the class just preceding the median class.
Standard Deviation	**For Ungrouped Data**	$S.D.(\sigma)=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}$ or $\sqrt{\frac{1}{n}\sum_{i=1}^{n}x_i^2-\bar{x}^2}$
	For Grouped Data	**For Discrete and Continuous Distribution** $\sigma=\sqrt{\frac{1}{N}\sum_{i=1}^{n}f_i(x_i-\bar{x})^2}$ or $\sigma=\frac{1}{N}\sqrt{N\sum_{i=1}^{n}f_ix_i^2-\left(\sum_{i=1}^{n}f_ix_i\right)^2}$ where x_i's are mid points of class intervals and $N=\sum_{i=1}^{n}f_i$
	Shortcut Method	$\sigma=\frac{h}{N}\sqrt{N\sum_{i=1}^{n}f_iy_i^2-\left(\sum_{i=1}^{n}f_iy_i\right)^2}$, where $y_i=\frac{x_i-A}{h}$

REMARKS

- Variance of a variate is the mean square of all the deviations of the values of the variate x from the mean of the observations and it is denoted by σ^2 or Var (x).

  Variance $= \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$

- Standard deviation is the positive square root of the variance and hence it is denoted by σ.

LIMITATIONS OF MEAN DEVIATION

(i) The mean deviation about median calculated for such series (where the degree of variability is very high) can not be fully relied.

(ii) The mean deviation about the mean is not very scientific since the sum of the deviations from the mean (if we ignored the minus signs) is more than the sum of the deviations from median.

(iii) The mean deviation is calculated on the basis of absolute values of the deviations and therefore, cannot be subjected to further algebraic treatment.

ANALYSIS OF FREQUENCY DISTRIBUTIONS

The measure of variability which is independent of units is called coefficient of variance and is denoted by $C.V.$

Thus, $C.V.=\frac{\sigma}{\bar{x}}\times 100,$ where $\bar{x}\neq 0$

- For two series with equal means, the series with greater standard deviation is more variable and the series with lesser value of standard deviation is more consistent.

Note :

For two groups of data taken together,

- $\bar{x}=\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}$
- $\sigma^2=\frac{1}{n_1+n_2}\left[n_1\sigma_1^2+n_2\sigma_2^2+\left(\frac{n_1n_2}{n_1+n_2}\right)(\bar{x}_1-\bar{x}_2)^2\right]$

RANDOM EXPERIMENT

If an experiment is performed many times under similar conditions and the outcome each time is not the same, then this experiment is called a random experiment.

- **Outcomes :** It is the possible result of the random experiment.
- **Sample Space :** It is the set of all possible outcomes of a random experiment. It is usually denoted by S.

EVENT

It is a subset of the sample space S.

Occurrence of an Event : For an event $E = \{p, q, r\}$, if $p \in E$, then it is said that event E occurred.

TYPES OF EVENTS

(1)	**Impossible and Sure Event :** The empty set ϕ is also a subset of sample space S and it represents an impossible event. Sample space S is also a subset of S, it represents a sure event.
(2)	**Simple Event :** An event is said to be a simple event if it is a singleton subset of the sample space S.
(3)	**Compound Event :** An event which is not a simple event is called compound event. In other words, an event is said to be compound event if it has more than one sample point.

ALGEBRA OF EVENTS

(1)	**Complementary Event :** Corresponding to every event A, we define an event "not A" which is said to occur when A does not occur. Thus, the event "not A" denoted by A' or $\bar{A}$ is called the complementary event of A.
(2)	**The Event 'A or B' :** Let A and B be two events associated with a sample space, then the event 'A or B' denoted by $A \cup B$ is the event 'either A or B or both'. $A \cup B = \{x : x \in A \text{ or } x \in B\}$
(3)	**The Event 'A and B' :** If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'. $A \cap B = \{x : x \in A \text{ and } x \in B\}$
(4)	**The Event 'A but not B' :** $A - B$ is the set of all those elements which are in A but not in B. Therefore, the set $A - B$ denotes the event 'A but not B'. We know that $A - B = A \cap B'$.

MUTUALLY EXCLUSIVE EVENTS

Two or more events are said to be mutually exclusive if one of them occurs, others cannot occur. Thus two or more events are said to be mutually exclusive if no two of them can occur together.

Thus events $A_1, A_2, ..., A_n$ are mutually exclusive if and only if $A_i \cap A_j = \phi$ for $i \neq j$.

EXHAUSTIVE EVENTS

For a random experiment, a set of events (cases) is said to be exhaustive if atleast one of them must necessarily happen every time the experiment is performed.

Thus events $A_1, A_2, ..., A_n$ are exhaustive if and only if $\bigcup_{i=1}^{n} A_n = S$, where S is sample space.

Note : If $E_i \cap E_j = \phi$ for $i \neq j$ *i.e.*, events E_i and E_j are pairwise disjoint and $\bigcup_{i=1}^{n} E_i = S$, then events $E_1, E_2, ..., E_n$ are called mutually exclusive and exhaustive events.

REMARKS

- For any event E, $P(E) \geq 0$.
- For any sample space, $P(S) = 1$
- Let S be a sample space and E be an event, such that $n(S) = l$ and $n(E) = m$. If each outcome is equally likely, then it follows that
$$P(E) = \frac{m}{l} = \frac{\text{Number of outcomes favourable to } E}{\text{Total possible outcomes}}$$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
If A and B are disjoint sets, *i.e.*, they are mutually exclusive events, then $A \cap B = \phi$.
Therefore $P(A \cap B) = P(\phi) = 0$
Thus, for mutually exclusive events A and B, we have $P(A \cup B) = P(A) + P(B)$
- $P(A') = P(\text{not } A) = 1 - P(A)$

PROBLEMS

Very Short Answer Type

1. Three coins are tossed once. Find the probability of getting at most two heads.
2. The probability of two events A and B are 0.21 and 0.53 respectively. The probability of their simultaneous occurrence is 0.18. Find the probability that neither A nor B occurs.
3. What is the probability that a randomly chosen two-digit integer is a multiple of 3?

4. The runs scored by two batsmen B_1 and B_2 in their last ten matches as given below :

Match	1	2	3	4	5	6	7	8	9	10
B_1	30	91	0	64	42	80	30	5	117	71
B_2	53	46	48	50	53	53	58	60	57	52

Find the range of the given distribution?

5. Given $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$. Find $P(A \text{ or } B)$, if A and B are mutually exclusive events.

Short Answer Type

6. Find the variance and standard deviation of the following data :

5, 9, 10, 12, 8, 13, 6

7. 4 cards are drawn from a well-shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade?

8. Find the mean deviation about the median for the data given below :

11, 3, 8, 7, 5, 14, 10, 2, 9

9. The variance of 'n' observation is σ_1^2. If each observation is multiplied by a, find the new variance.

10. In an essay competition, the odds in favour of competitors P, Q, R, S are 1 : 2, 1 : 3, 1 : 4 and 1 : 5 respectively. Find the probability that one of them wins the competition.

Long Answer Type - I

11. A box contains 4 red, 5 white and 6 black balls. A person draws 4 balls from the box at random. Find the probability of selecting at least one ball of each colour.

12. A coin is tossed. If head comes up, a die is thrown but if tail comes up, the coin is tossed again. Find the probability of obtaining :

(i) two tails (ii) head and number 6

(iii) head and an even number

13. The mean and variance of six observations are 8 and 16 respectively. If each observation is multiplied by 3, find the new mean and new variance of the resulting observations.

14. Let $x_1, x_2, x_3, \ldots, x_n$ be n values of a variable X, and let $x_i = a + hu_i$, $i = 1, 2, \ldots, n$, where $u_1, u_2, \ldots, u_n$ are the values of variable U. Then, prove that $\text{Var}(X) = h^2\,\text{Var}(U)$, $h \neq 0$.

15. Four cards are drawn at a time from a pack of 52 playing cards. Find the probability of getting all four cards of the same suit.

Long Answer Type - II

16. The mean and standard deviation of 20 observations are bound to be 10 and 2 respectively. On rechecking, it was found that an observation 8 was incorrect. Calculate the correct mean and standard deviation in each of the following cases :

(i) If the wrong observation is omitted.

(ii) If it is replaced by 12.

17. Three dice are thrown simultaneously. Find the probability that :

(i) all of them show the same face.

(ii) all show distinct faces.

(iii) two of them show the same face.

18. In a single throw of two dice, find the probability that neither a doublet nor a total of 10 will appear.

19. Find the mean and standard deviation of first n terms of an A.P. whose first term is a and common difference is d.

20. A box contains 100 bolts and 50 nuts. It is given that 50% bolts and 50% nuts are rusted. Two objects are selected from the box at random. Find the probability that either both are bolts or both are rusted.

SOLUTIONS

1. At most two heads can be obtained in any one of the following ways

$HHT, THH, HTH, HTT, THT, TTH, TTT$

$\therefore$ Favourable number of elementary events = 7.

$\therefore$ Required probability $= \frac{7}{8}$

2. Here $P(A) = 0.21$, $P(B) = 0.53$ and $P(A \cap B) = 0.18$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.21 + 0.53 - 0.18 = 0.56$

Now $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$= 1 - 0.56 = 0.44$

3. Two-digit positive integers are 10, 11, 12 ..... , 98, 99. These are 90 numbers out of which one number can be chosen in 90 ways.

$\therefore$ Total number of elementary events = 90.

Out of these 90 numbers, 30 numbers (12, 15, 18, ... 96, 99) are multiples of 3. One number out of these 30 numbers can be chosen in 30 ways.

$\therefore$ Favourable number of elementary events = 30

Hence, required probability $=\frac{30}{90}=\frac{1}{3}$.

4. Range of a distribution
$=$ Maximum value – Minimum value
$\therefore$ Range of scores of batsmen $B_1 = 117 - 0 = 117$
Range of scores of batsmen $B_2 = 60 - 46 = 14$

5. Here $P(A)=\frac{3}{5},\ P(B)=\frac{1}{5}$

Since A and B are mutually exclusive events.

$\therefore\ P(A \cup B) = P(A) + P(B)$

$\therefore\ P(A \cup B) = \frac{3}{5}+\frac{1}{5}=\frac{4}{5}$.

6. We have,

Variance $(\sigma^2)=\frac{1}{n}\sum x_i^2 - \bar{x}^2$

Here n = number of observations = 7

and $\bar{x}=\frac{\sum x_i}{n}=\frac{5+9+10+12+8+13+6}{7}=9$

$\sum x_i^2 = 5^2 + 9^2 + 10^2 + 12^2 + 8^2 + 13^2 + 6^2 = 619$

$\therefore\ \sigma^2 = \frac{1}{7}\cdot 619 - 9^2 = 7.43$ and $\sigma = \sqrt{7.43} = 2.72$

Thus, variance = 7.43 and standard deviation = 2.72.

7. From a pack of 52 cards, 4 cards can be drawn in $^{52}C_4$ ways.
There are 13 cards of diamond and 13 cards of spades.
Now 3 cards of diamond out of 13 cards of diamond can be drawn in $^{13}C_3$ ways and 1 card of spade out of 13 cards of spade can be drawn in $^{13}C_1$ ways.
Thus the probability of obtaining 3 diamond and

1 spade card $=\frac{^{13}C_3 \times {}^{13}C_1}{^{52}C_4}=\frac{286}{20825}$

8. Arranging the given data in an ascending order, we get :

2, 3, 5, 7, 8, 9, 10, 11, 14

Here $n = 9$, which is odd.

$\therefore$ Median $=\frac{1}{2}(n+1)^{\text{th}}$ observation

$=\frac{1}{2}(9+1)^{\text{th}}$ observation $= 5^{\text{th}}$ observation $= 8$

Thus, median $(M) = 8$.

$\therefore\ \sum_{i=1}^{9}|x_i - M| = (6+5+3+1+0+1+2+3+6)=27$

$\Rightarrow\ MD(M)=\frac{\sum_{i=1}^{9}|x_i - M|}{9}=\frac{27}{9}=3$

Hence, $MD(M) = 3$.

9. Let the 'n' observations be $x_1, x_2, \ldots, x_n$ and $\bar{x}_1$ be their mean.

Then, variance $(\sigma_1^2)=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x}_1)^2$ ...(i)

If each observation is multiplied by a, the new mean, $\bar{x}_2$ is given by

$$\bar{x}_2 = \frac{1}{n}\sum_{i=1}^{n}(ax_i)=\frac{a}{n}\sum_{i=1}^{n}x_i = a\bar{x}_1 \quad \text{...(ii)}$$

$\therefore$ The new variance, $\sigma_2^2(\text{say})=\frac{1}{n}\sum_{i=1}^{n}(ax_i-\bar{x}_2)^2$

$=\frac{1}{n}\sum_{i=1}^{n}a^2(x_i-\bar{x}_1)^2$ [From (ii)]

$=a^2\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x}_1)^2$

$\therefore\ \sigma_2^2 = a^2\sigma_1^2$

Thus, the new variance is $a^2\sigma_1^2$.

10. Let A, B, C, D be the events that the competitors P, Q, R and S respectively win the competition. Then,

$P(A)=\frac{1}{3}, P(B)=\frac{1}{4}, P(C)=\frac{1}{5}$ and $P(D)=\frac{1}{6}$

Since only one competitor can win the competition. Therefore, A, B, C, D are mutually exclusive events.

$\therefore$ Required probability $= P(A \cup B \cup C \cup D)$

$= P(A) + P(B) + P(C) + P(D)$ [By addition theorem]

$=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{19}{20}$

11. Here random experiment is drawing 4 balls out of 4 red, 5 white and 6 black balls.
Let S be the sample space.
$\therefore\ n(S)$ = number of ways of selecting 4 balls out of 15 balls $= {}^{15}C_4$
Selection of balls such that there is at least one ball of each colour can be done by the following mutually exclusive ways :
(i) 1 red, 1 white and 2 black balls
(ii) 1 red, 2 white and 1 black ball
(iii) 2 red, 1 white and 1 black ball
Let A = Event that 1 red, 1 white and 2 black balls are drawn.
B = Event that 1 red, 2 white and 1 black ball are drawn.
C = Event that 2 red, 1 white and 1 black ball are drawn.
Here A, B and C are mutually exclusive events.

Hence required probability = $P(A \cup B \cup C)$
$= P(A) + P(B) + P(C)$

$$= \frac{{}^4C_1 \cdot {}^5C_1 \cdot {}^6C_2 \cdot}{{}^{15}C_4} + \frac{{}^4C_1 \cdot {}^5C_2 \cdot {}^6C_1}{{}^{15}C_4} + \frac{{}^4C_2 \cdot {}^5C_1 \cdot {}^6C_1}{{}^{15}C_4} = \frac{48}{91}$$

12. The samples space S associated with the given random experiment is

$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, H), (T, T)\}$

$\therefore$ Total number of elementary events = 8

(i) Two tails are obtained only in one way, *i.e.*, (T, T)

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= \frac{1}{8}$.

(ii) Head and the number 6 is obtained in only one way *i.e.*, when the outcome is $(H, 6)$

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= \frac{1}{8}$.

(iii) Head and an even number can be obtained in any one of the following ways :

$(H, 2), (H, 4), (H, 6)$

$\therefore$ Favourable number of elementary events = 3

Hence, required probability $= \frac{3}{8}$.

13. Let the given observations be $x_1, x_2, x_3, x_4, x_5, x_6$.

Since, mean = 8 $\Rightarrow \frac{1}{6}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) = 8$

$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 48$ ...(i)

Also, variance = 16

$$\Rightarrow \frac{1}{6}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) - 8^2 = 16$$

$$\left[\because \sigma^2 = \frac{\Sigma x_i^2}{n} - (\bar{x})^2\right]$$

$\Rightarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 480$ ...(ii)

When each observation is multiplied by 3, then new observations are $3x_1, 3x_2, 3x_3, 3x_4, 3x_5$ and $3x_6$.

$\therefore$ New mean $= \frac{1}{6}(3x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 + 3x_6)$

$$= \frac{3}{6}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) = \left(\frac{1}{2} \times 48\right) = 24$$

[using (i)]

$$\therefore \text{ New variance} = \frac{(3x_1)^2 + (3x_2)^2 + (3x_3)^2 + (3x_4)^2 + (3x_5)^2 + (3x_6)^2}{6} - (24)^2$$

$$= \frac{9}{6}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) - 576$$

$$= \left(\frac{9}{6} \times 480\right) - 576 = (720 - 576) = 144 \text{ [using (ii)]}$$

Hence, new mean = 24 and new variance = 144.

14. We have,

$x_i = a + hu_i, i = 1, 2, \ldots, n$

$$\Rightarrow \sum_{i=1}^{n} x_i = \sum_{i=1}^{n}(a + hu_i)$$

$$\Rightarrow \sum_{i=1}^{n} x_i = na + h \sum_{i=1}^{n} u_i$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} x_i = a + h\left(\frac{1}{n}\sum_{i=1}^{n} u_i\right)$$

$$\Rightarrow \bar{x} = a + h\,\bar{u} \quad \left[\because \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } \bar{u} = \frac{1}{n}\sum_{i=1}^{n} u_i\right]$$

$$\therefore x_i - \bar{x} = (a + h\,u_i) - (a + h\,\bar{u}),\ i = 1, 2, \ldots, n$$

$$\Rightarrow x_i - \bar{x} = h\,(u_i - \bar{u}),\ i = 1, 2, \ldots, n$$

$$\Rightarrow (x_i - \bar{x})^2 = h^2(u_i - \bar{u})^2,\ i = 1, 2, \ldots, n$$

$$\Rightarrow \sum_{i=1}^{n}(x_i - \bar{x})^2 = h^2\sum_{i=1}^{n}(u_i - \bar{u})^2$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = h^2\left\{\frac{1}{n}\sum_{i=1}^{n}(u_i - \bar{u})^2\right\}$$

(Dividing both sides by n)

$\Rightarrow$ Var $(X) = h^2$ Var (U).

15. Here random experiment is drawing 4 cards from a pack of 52 playing cards.

Let S be the sample space.

$\therefore n(S)$ = number of ways of drawing 4 cards from a pack of 52 cards $= {}^{52}C_4$

Let A = Event that all cards are spade cards,

B = Event that all cards are heart cards,

C = Event that all cards are diamond cards,

and D = Event that all cards are club cards.

Then A, B, C and D are mutually exclusive events.

$\therefore n(A) = {}^{13}C_4, n(B) = {}^{13}C_4, n(C) = {}^{13}C_4$ and $n(D) = {}^{13}C_4$

$$\therefore P(A) = \frac{{}^{13}C_4}{{}^{52}C_4},\ P(B) = \frac{{}^{13}C_4}{{}^{52}C_4},\ P(C) = \frac{{}^{13}C_4}{{}^{52}C_4},$$

and $P(D) = \frac{{}^{13}C_4}{{}^{52}C_4}$

Now, required probability = $P(A \cup B \cup C \cup D)$

$= P(A) + P(B) + P(C) + P(D)$

$$= \frac{4(^{13}C_4)}{^{52}C_4} = \frac{44}{4165}$$

16. We have, $n = 20$, $\bar{x} = 10$ and $\sigma = 2$

$$\therefore \quad \bar{x} = \frac{1}{n}\sum x_i \Rightarrow \sum x_i = n\bar{x} = 20 \times 10 = 200$$

$\therefore$ Incorrect $\sum x_i = 200$

and $\sigma = 2 \Rightarrow \sigma^2 = 4$

$$\Rightarrow \frac{1}{n}\sum x_i^2 - \bar{x}^2 = 4 \Rightarrow \frac{1}{20}\sum x_i^2 - 100 = 4$$

$$\Rightarrow \sum x_i^2 = 2080$$

$\therefore$ Incorrect $\sum x_i^2 = 2080$

(i) If we omit the wrong item, 8 from the observations, then 19 observations are left.

Correct $\sum x_i + 8 =$ Incorrect $\sum x_i$

$\therefore$ Correct $\sum x_i = 200 - 8 = 192$

$\therefore$ Correct mean $= \frac{192}{19} = 10.105$

and Correct $\sum x_i^2 + 8^2 =$ Incorrect $\sum x_i^2$

$\therefore$ Correct $\sum x_i^2 = 2080 - 64 = 2016$

$\Rightarrow$ Correct variance

$$= \frac{1}{19}(\text{Correct} \sum x_i^2) - (\text{Correct mean})^2$$

$$= \frac{2016}{19} - \left(\frac{192}{19}\right)^2 = \frac{1440}{361}$$

$\therefore$ Correct standard deviation $= \sqrt{\frac{1440}{361}} = 1.997$

(ii) If we replace the wrong observation by 12.

Incorrect $\sum x_i - 8 + 12 =$ Correct $\sum x_i$

$\Rightarrow$ Correct $\sum x_i = 200 + 4 = 204$

and Correct mean $= \frac{204}{20} = 10.2$

Also, Incorrect $\sum x_i^2 - 8^2 + 12^2 =$ Correct $\sum x_i^2$

$\therefore$ Correct $\sum x_i^2 = 2080 - 8^2 + 12^2 = 2160$

$\Rightarrow$ Correct variance $= \frac{1}{20}(\text{Correct} \sum x_i^2) - (\text{Correct mean})^2$

$$= \frac{2160}{20} - \left(\frac{204}{20}\right)^2 = \frac{99}{25}$$

$\therefore$ Correct standard deviation $= \sqrt{\frac{99}{25}} = 1.9899$

17. The total number of elementary events associated to the random experiment of throwing three dice simultaneously is $6 \times 6 \times 6 = 6^3$.

(i) All dice show the same face in one of the following mutually exclusive ways :

(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6)

So, favourable number of elementary events = 6.

Hence, required probability $= \frac{6}{6^3} = \frac{1}{36}$

(ii) The total number of ways in which all dice show different faces is equal to the number of ways of arranging 6 distinct objects by taking three at a time *i.e.*, $^6C_3 \times 3!$.

So, favourable number of elementary events $= {^6C_3} \times 3!$

Hence, required probability $= \frac{^6C_3 \times 3!}{6^3} = \frac{5}{9}$

(iii) Select a number which occurs on two dice out of the six numbers 1, 2, 3, 4, 5, 6 marked on the six faces of a die. This can be done in 6C1 ways. Now, we have three numbers like 1, 1, 2 ; 2, 2, 5 etc. These three digits can be arranged in $\frac{3!}{2!}$ ways.

So, the favourable number of elementary events $= {^6C_1} \times {^5C_1} \times \frac{3!}{2!}$

Hence, required probabitity $= \frac{90}{6^3} = \frac{5}{12}$

18. Let S be the sample space. Then, $n(S) = 36$.

Let E_1 = Event that a doublet appears,

and E_2 = Event of getting a total of 10.

Then, $E_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$,

and $E_2 = \{(4, 6), (5, 5), (6, 4)\}$.

$\therefore \quad (E_1 \cap E_2) = \{(5, 5)\}$

Thus, $n(E_1) = 6$, $n(E_2) = 3$ and $n(E_1 \cap E_2) = 1$

$$\therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{36} = \frac{1}{6}, P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

and $P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)} = \frac{1}{36}$

$\therefore$ P(getting a doublet or a total of 10)

$= P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \left(\frac{1}{6} + \frac{1}{12} - \frac{1}{36}\right) = \frac{8}{36} = \frac{2}{9}$$

$\therefore$ P(getting neither a doublet nor a total of 10)

$= P(\bar{E}_1 \text{ and } \bar{E}_2) = P(\bar{E}_1 \cap \bar{E}_2)$

$$= P(\overline{E_1 \cup E_2}) = 1 - P(E_1 \cup E_2) = \left(1 - \frac{2}{9}\right) = \frac{7}{9}$$

Hence, the required probability is $\frac{7}{9}$.

19. The terms of the A.P. are : $a, a + d, a + 2d, a + 3d, ..., a + (r - 1) d, ... , a + (n - 1) d$. Let $\bar{x}$ be the mean of these terms. Then,

$$\bar{x} = \frac{1}{n}\{a + (a + d) + (a + 2d) + ... + (a + (n - 1)d)\}$$

$$= \frac{1}{n}\left[\frac{n}{2}\{2a+(n-1)d\}\right] = a+(n-1)\frac{d}{2}$$

Let σ be the standard deviation of n terms of the A.P. then,

$$\sigma^2 = \frac{1}{n}\sum_{r=1}^{n}[\{a+(r-1)d\}-\bar{x}]^2$$

$$\Rightarrow \sigma^2 = \frac{1}{n}\sum_{r=1}^{n}\left[\{a+(r-1)d\}-\left\{a+(n-1)\frac{d}{2}\right\}\right]^2$$

$$\Rightarrow \sigma^2 = \frac{d^2}{4n}\left[\sum_{r=1}^{n}(2r-2-n+1)^2\right]$$

$$\Rightarrow \sigma^2 = \frac{d^2}{4n}\left[\sum_{r=1}^{n}\{2r-(n+1)\}^2\right]$$

$$\Rightarrow \sigma^2 = \frac{d^2}{4n}\left[\sum_{r=1}^{n}\{4r^2-4(n+1)r+(n+1)^2\}\right]$$

$$\Rightarrow \sigma^2 = \frac{d^2}{4n}\left[4\left(\sum_{r=1}^{n}r^2\right)-4(n+1)\left(\sum_{r=1}^{n}r\right)+\sum_{r=1}^{n}(n+1)^2\right]$$

$$= \frac{d^2}{4n}\left\{\frac{4n(n+1)(2n+1)}{6}-\frac{4(n+1)n(n+1)}{2}+n(n+1)^2\right\}$$

$$\Rightarrow \sigma^2 = \frac{d^2}{4n}\left\{\frac{2n(n+1)(2n+1)}{3}-n(n+1)^2\right\}$$

$$\Rightarrow \sigma^2 = \frac{d^2}{12n}n(n+1)\{2(2n+1)-3(n+1)\} = \frac{(n^2-1)d^2}{12}$$

$$\Rightarrow \sigma = d\sqrt{\frac{n^2-1}{12}}$$

20. Total number of objects = (100 + 50) = 150

Let S be the sample space. Then,

$n(S)$ = Number of ways of selecting 2 objects out of 150 $= {}^{150}C_2$

Number of rusted objects

= (50% of 100) + (50% of 50) = (50 + 25) = 75

Let E_1 = Event of selecting 2 bolts out of 100 bolts and E_2 = Event of selecting 2 rusted objects out of 75 rusted objects.

$\therefore$ $(E_1 \cap E_2)$ = Event of selecting 2 rusted bolts out of 50 rusted bolts

$\therefore$ $n(E_1)$ = Number of ways of selecting 2 bolts out of 100 $= {}^{100}C_2$

$\therefore$ $n(E_2)$ = Number of ways of selecting 2 rusted objects out of 75 $= {}^{75}C_2$

$\therefore$ $n(E_1 \cap E_2)$ = Number of ways of selecting 2 rusted bolts out of 50 $= {}^{50}C_2$

$$\therefore\ P(E_1) = \frac{n(E_1)}{n(S)} = \frac{{}^{100}C_2}{{}^{150}C_2},\ P(E_2) = \frac{n(E_2)}{n(S)} = \frac{{}^{75}C_2}{{}^{150}C_2}$$

$$\text{and } P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)} = \frac{{}^{50}C_2}{{}^{150}C_2}$$

P(selecting both bolts or both rusted objects)

$= P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \frac{{}^{100}C_2}{{}^{150}C_2} + \frac{{}^{75}C_2}{{}^{150}C_2} - \frac{{}^{50}C_2}{{}^{150}C_2} = \frac{({}^{100}C_2 + {}^{75}C_2 - {}^{50}C_2)}{{}^{150}C_2}$$

$$= \frac{(4950+2775-1225)}{11175} = \frac{6500}{11175} = \frac{260}{447}$$

Hence, the required probability is $\frac{260}{447}$.

MPP-7 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer.

Self check table given at the end will help you to check your readiness.

Straight Lines | Conic Sections

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. Given a family of lines $a(2x + y + 4) + b(x - 2y - 3) = 0$, the number of lines belonging to the family at a distance $\sqrt{10}$ from the point $A(2, -3)$ is

(a) 3 (b) 2
(c) 1 (d) None of these

2. If $\left(m_i, \frac{1}{m_i}\right)$, $i = 1, 2, 3, 4$ are four distinct points on the circle $x^2 + y^2 = a^2$ then value of $m_1 + m_2 + m_3 + m_4$ equals

(a) 0 (b) -1 (c) 1 (d) $-a^2$

3. Two circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$ are given. Then the equation of the circle through their points of intersection and the point $(1, 1)$ is

(a) $x^2 + y^2 - 6x + 4 = 0$
(b) $x^2 + y^2 - 3x + 1 = 0$
(c) $x^2 + y^2 - 4x + 2 = 0$
(d) None of these

4. A tangent to the ellipse $4x^2 + 9y^2 = 36$ is cut by the tangent at the extremities of the major axis at T and T'. The circle on TT' as diameter passes through the point is

(a) $(\pm\sqrt{5}, 0)$ (b) $(5, 0)$
(c) $(0, 0)$ (d) $(3, 2)$

5. The equation of parabola whose length of latus rectum is 2 units, axis of line is $x + y - 2 = 0$ and tangent at the vertex is $x - y + 4 = 0$ is given by

(a) $(x + y - 2)^2 = 4\sqrt{2}\,(x - y + 4)^2$
(b) $(x - y - 4)^2 = 4\sqrt{2}\,(x + y - 2)$
(c) $(x + y - 2)^2 = 2\sqrt{2}\,(x - y + 4)$
(d) None of these

6. If e_1 is the eccentricity of the conic $9x^2 + 4y^2 = 36$ and e_2 is the eccentricity of the conic $9x^2 - 4y^2 = 36$ then which is true ?

(a) $e_1^2 + e_2^2 = 2$ (b) $3 < e_1^2 + e_2^2 < 4$
(c) $e_1^2 + e_2^2 > 4$ (d) None of these

One or More Than One Option(s) Correct Type

7. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are

(a) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (b) $\left(\frac{-2}{5}, \frac{1}{5}\right)$
(c) $\left(\frac{-2}{5}, \frac{-1}{5}\right)$ (d) $\left(\frac{2}{5}, \frac{-1}{5}\right)$

8. The product of intercepts made by the line $x \tan\alpha + y \sec\alpha = 1$ on the coordinate axes is equal to $\sin\alpha$, then value of α is

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$
(c) $\frac{5\pi}{4}$ (d) $\frac{7\pi}{4}$

9. The set of real values of a for which at least one tangent to the parabola $y^2 = 4ax$ becomes normal to the circle $x^2 + y^2 - 2ax - 4ay + 3a^2 = 0$, is

(a) $[1, 2]$ (b) $[\sqrt{2}, 3]$
(c) R (d) ϕ

10. If the points $\left(\frac{a^3}{a-1}, \frac{a^2-3}{a-1}\right), \left(\frac{b^3}{b-1}, \frac{b^2-3}{b-1}\right)$ and $\left(\frac{c^3}{c-1}, \frac{c^2-3}{c-1}\right)$ where a, b, c are different from 1, lie on the line $lx + my + n = 0$, then
(a) $a + b + c = -m/l$
(b) $ab + bc + ca = n/l$
(c) $abc = (3m + n)/l$
(d) $abc - (bc + ca + ab) + 3(a + b + c) = 0$

11. The equation of a circle in which the chord joining the points (1, 2) and (2, –1) subtends an angle of $\pi/4$ at any point on the circumference is
(a) $x^2 + y^2 - 5 = 0$
(b) $x^2 + y^2 - 6x - 2y + 5 = 0$
(c) $x^2 + y^2 + 6x + 2y - 15 = 0$
(d) $x^2 + y^2 - 2x - 4y + 4 = 0$

12. If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes in G and g and C is the centre of the hyperbola, then
(a) $PG = PC$ (b) $Pg = PC$
(c) $PG = Pg$ (d) $Gg = 2PC$

13. If the slopes of the lines $3x^2 + 2hxy + 4y^2 = 0$ are in the ratio 3 : 1, then h equals
(a) 4 (b) – 4
(c) $-\frac{1}{2}$ (d) $-\frac{1}{4}$

Comprehension Type

The equation of the curve represented by
$C \equiv 9x^2 - 24xy + 16y^2 - 20x - 15y - 60 = 0$, then

14. The equation of the axis of the curve C is
(a) $x = 4y$ (b) $3x = y$
(c) $3x = -4y$ (d) $3x = 4y$

15. The equation of the directrix of the curve C is
(a) $16x + 9y = -53$ (b) $16x + 12y + 53 = 0$
(c) $16x + 12y = 53$ (d) $16x + 9y - 53 = 0$

Matrix Match Type

16. For the circle $x^2 + y^2 + 4x + 6y - 19 = 0$.
Match the following:

	Column I		Column II
P.	Length of the tangent from (6, 4) to the circle is	1.	$\frac{72\sqrt{226}}{113}$
Q.	Length of the chord of contact from (6, 4) to the circle is	2.	$\sqrt{113}$
R.	Distance of (6, 4) from the centre of the circle is	3.	$\sqrt{113} - \sqrt{32}$
S.	Shortest distance of (6, 4) from the circle is	4.	9

	P	Q	R	S
(a)	1	2	3	4
(b)	1	3	4	2
(c)	4	1	2	3
(d)	3	1	2	4

Integer Answer Type

17. If the distance of the point (2, 3) from the line $2x - 3y + 9 = 0$ measured along the line $2x - 2y + 5 = 0$ is denoted by r, then $[r]$ is (where $[\cdot]$ denotes greatest integer function)

18. The number of points on the circle $2x^2 + 2y^2 - 3x = 0$ which are at a distance 2 from the point (– 2, 1) is

19. The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide then $b^2 =$

20. The product of the perpendiculars drawn from the two foci of an ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ to the tangent at any point of the ellipse is

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ......
No. of questions correct ......
Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT BOOSTERS

Class XII

Vectors and 3D Geometry

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

DEFINITIONS

- A vector may be described as a quantity having both magnitude and direction. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the initial point and B is called the terminal point. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.
- **Zero vector :** A vector of zero magnitude *i.e.* which has the same initial and terminal point, is called a zero vector. It is denoted by $\vec{0}$.
- **Unit vector :** A vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector and is denoted by $\hat{a}$. Symbolically $\hat{a}=\dfrac{\vec{a}}{|\vec{a}|}$.
- **Equal vectors :** Two vectors are said to be equal, if they have the same magnitude, direction and represent the same physical quantity.
- **Collinear vectors :** Two vectors are said to be collinear, if their directed line segments are parallel disregards to their direction. Collinear vectors are also called parallel vectors. If they have the same direction they are named as like vectors otherwise unlike vectors.

  Symbolically, two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$.
- **Coplanar vectors :** A given number of vectors are called coplanar if their all line segments are parallel to the same plane.
- **Position vector :** Let O (origin) be a fixed point, then the position vector of a point P is the vector $\overrightarrow{OP}$.

  If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B respectively, then

  $\overrightarrow{AB} = \vec{b} - \vec{a}$ = P.V. of B – P.V. of A

VECTOR ADDITION

- Let $\vec{a}$, b and $\vec{c}$ be the position vectors of $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively. Then, by triangle law $\vec{c} = \vec{a} + \vec{b}$ where OC is the diagonal of the parallelogram $OACB$.
- $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ (commutative)
- $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ (associative)
- $\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$
- $\vec{a}+(-\vec{a}) = \vec{0} = (-\vec{a})+\vec{a}$

MULTIPLICATION OF VECTOR BY SCALARS

If $\vec{a}$ is a vector and m is a scalar, then $m\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times to that of $\vec{a}$. This multiplication is called scalar multiplication. If $\vec{a}$ and $\vec{b}$ are vectors and m, n are scalars, then

$$m(\vec{a}) = (\vec{a})m = m\vec{a}$$
$$m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$$
$$(m+n)\vec{a} = m\vec{a} + n\vec{a}$$
$$m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$$

SECTION FORMULA

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the position vector of a point which divides AB in the ratio $m : n$ is given by $\vec{r} = \dfrac{n\vec{a}+m\vec{b}}{m+n}$.

Note : Position vector of mid-point of $AB = \dfrac{\vec{a}+\vec{b}}{2}$.

DIRECTION COSINES

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and α, β, γ are the angles which the vector makes with +ve directions of *OX*, *OY* and *OZ* called direction angles, then their cosines are called the direction cosines.

i.e., $\cos\alpha = \frac{a_1}{|\vec{a}|}$, $\cos\beta = \frac{a_2}{|\vec{a}|}$, $\cos\gamma = \frac{a_3}{|\vec{a}|}$

Note : $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

VECTOR EQUATION OF A LINE

- Parametric vector equation of a line passing through two points $A(\vec{a})$ and $B(\vec{b})$ is given by, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$, where t is a parameter.
- If the line passes through the point $A(\vec{a})$ and is parallel to the vector $\vec{b}$, then its equation is, $\vec{r} = \vec{a} + t\vec{b}$.
- The equations of the bisectors of the angle between the lines, $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$ is $\vec{r} = \vec{a} + t(\vec{b} + \vec{c})$ and $\vec{r} = \vec{a} + p(\vec{c} - \vec{b})$

TEST OF COLLINEARITY

Three points *A*, *B*, *C* with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if and only if there exist scalars *x*, *y*, *z* not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} = 0$, where $x + y + z = 0$.

SCALAR PRODUCT OF TWO VECTORS

- $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta \ (0 \le \theta \le \pi)$,
  Note that if θ is acute, then $\vec{a}\cdot\vec{b} > 0$ and if θ is obtuse, then $\vec{a}\cdot\vec{b} < 0$
- $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)
- $\vec{a}\cdot(\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (distributive)
- $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b} \quad (\vec{a} \ne 0, \vec{b} \ne 0)$
- $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$; $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

REMARKS

- The angle ϕ between $\vec{a}$ and $\vec{b}$ is given by $\cos\phi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}, 0 \le \phi \le \pi$.
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then
  $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
  $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$.
- Maximum value of $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$.
- Minimum value of $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$.
- Any vector $\vec{a}$ can be written as ,
  $\vec{a} = (\vec{a}\cdot\hat{i})\hat{i} + (\vec{a}\cdot\hat{j})\hat{j} + (\vec{a}\cdot\hat{k})\hat{k}$.
- A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$.
  Hence, bisector of the interior angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\lambda(\hat{a} + \hat{b})$, where $\lambda \in R^+$.
  Bisector of the exterior angle between two vectors $\vec{a}$ and $\vec{b}$ is $\lambda(\hat{a} - \hat{b}), \lambda \in R^+$.

VECTOR PRODUCT OF TWO VECTORS

- If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them, then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}$ and $\hat{n}$ forms a right handed screw system .
- **Lagrange's identity :** For any two vectors $\vec{a}$ and $\vec{b}$;
  $$(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$$
- **Formulation of vector product in terms of scalar product**
  The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that
  (i) $|\vec{c}| = \sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2}$
  (ii) $\vec{c}\cdot\vec{a} = 0, \vec{c}\cdot\vec{b} = 0$
  (iii) $\vec{a}, \vec{b}, \vec{c}$ form a right handed system
- $\vec{a}\times\vec{b} = 0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a} \ne 0, \vec{b} \ne 0)$ *i.e.* $\vec{a} = K\vec{b}$, where *K* is a scalar.
- $\vec{a}\times\vec{b} \ne \vec{b}\times\vec{a}$ (not commutative)
- $(m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b}) = m(\vec{a}\times\vec{b})$, where *m* is a scalar.
- $\vec{a}\times(\vec{b} + \vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ (distributive)
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$
- $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,
  then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

- Geometrically $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.
- Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\hat{n} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$
- A vector of magnitude r and perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$.
- If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

AREA OF TRIANGLE

- If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of three points A, B and C, then the area of triangle $ABC = \dfrac{1}{2}\left[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right]$.

  The points A, B and C are collinear if $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = 0$
- Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$.

SCALAR TRIPLE PRODUCT

- The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as

  $(\vec{a}\times\vec{b})\cdot\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$

  where θ is the angle between $\vec{a}$ and $\vec{b}$ and ϕ is the angle between $(\vec{a}\times\vec{b})$ and $\vec{c}$. It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product.
- Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}$, $\vec{b}$ and $\vec{c}$ *i.e.* $V = [\vec{a}\ \vec{b}\ \vec{c}]$
- In a scalar triple product, the position of dot and cross can be interchanged

  i.e. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

  or $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$
- $\vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b})$
- If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$; $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- Vectors $\vec{a}, b, \vec{c}$ are coplanar $\Leftrightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.
- Scalar product of three vectors, two of which are equal or parallel is 0.

REMARKS

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system and $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

- $[\hat{i}\ \hat{j}\ \hat{k}] = 1$
- $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$
- $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$
- The volume of the tetrahedron $OABC$ with O as origin and the position vectors of A, B and C being $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively is given by $V = \dfrac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$.
- The position vector of the centroid of a tetrahedron if the position vectors of its angular vertices are $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are given by $\dfrac{1}{4}[\vec{a}+\vec{b}+\vec{c}+\vec{d}]$.

VECTOR TRIPLE PRODUCT

Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then the expression $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector and is called a vector triple product.

Geometrical Interpretation of $\vec{a}\times(\vec{b}\times\vec{c})$:

Consider the expression $\vec{a}\times(\vec{b}\times\vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ and $(\vec{b}\times\vec{c})$. Now $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector perpendicular to the plane containing $\vec{a}$ and $(\vec{b}\times\vec{c})$ but $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$, therefore $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector lies in the plane of $\vec{b}$ and $\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a}\times(\vec{b}\times\vec{c})$ in terms of $\vec{b}$ and $\vec{c}$.

i.e. $\vec{a}\times(\vec{b}\times\vec{c}) = x\vec{b} + y\vec{c}$, where x and y are scalars.

- $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$
- $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$
- $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

LINEAR COMBINATIONS

- Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots$, then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, \ldots$ for any $x, y, z, \ldots \in R$.

- **Fundamental theorem in plane :** Let $\vec{a}, \vec{b}$ be non-zero, non-collinear vectors. Then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ *i.e.* there exist some unique $x, y \in R$ such that $x\vec{a} + y\vec{b} = \vec{r}$.
- **Fundamental theorem in space :** Let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$ can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ *i.e.* there exist some unique $x, y, z \in R$ such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.
- **Linearly independent and dependent vectors**

  (i) If $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are n non-zero vectors, and $k_1, k_2, ..., k_n$ are n scalars and if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + ... + k_n\vec{x}_n = 0$
  $\Rightarrow k_1 = 0, k_2 = 0, ..., k_n = 0$
  then we say that vectors $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are linearly independent vectors.

  (ii) If $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are not linearly independent, then they are said to be linearly dependent vectors. *i.e.* if $k_1\vec{x}_1 + k_2\vec{x}_2 + ... + k_n\vec{x}_n = 0$ and if there exists at least one $k_r \neq 0$, then $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are said to be linearly dependent vectors.

REMARKS

- If $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$, then $\vec{a}$ is expressed as a linear combination of vectors $\hat{i}, \hat{j}, \hat{k}$. Also, $\vec{a}, \hat{i}, \hat{j}, \hat{k}$ form a linearly dependent set of vectors. In general, every set of four vectors forms a linearly dependent system.
- $\hat{i}, \hat{j}, \hat{k}$ are linearly independent set of vectors. For $K_1\hat{i} + K_2\hat{j} + K_3\hat{k} = 0 \Rightarrow K_1 = 0 = K_2 = K_3$.
- Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent *i.e.*, $\vec{a}$ is parallel to $\vec{b}$ *i.e.* $\vec{a} \times \vec{b} = 0$. Conversely if $\vec{a} \times \vec{b} \neq 0$ then $\vec{a}$ and $\vec{b}$ are linearly independent.

COPLANARITY OF VECTORS

- If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar *i.e.*, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$. Conversely, if $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$, then the vectors are linearly independent.
- Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ where $x + y + z + w = 0$.

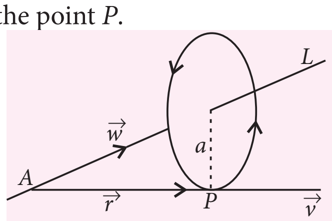
RECIPROCAL SYSTEM OF VECTORS

- $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non-coplanar vectors such that $\vec{a} \cdot \vec{a}' = \vec{b} \cdot \vec{b}' = \vec{c} \cdot \vec{c}' = 1$, then the two systems are called reciprocal system of vectors.

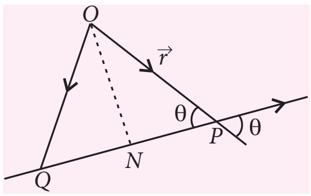
  Note : $\vec{a}' = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{b}' = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{c}' = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

APPLICATION OF VECTORS

- Work done against a constant force $\vec{F}$ over a displacement $\vec{s}$ is defined as $\vec{W} = \vec{F} \cdot \vec{s}$.
- The tangential velocity $\vec{v}$ of a body moving in a circle is given by $\vec{v} = \vec{w} \times \vec{r}$, where $\vec{r}$ is the position vector of the point P.



- The moment of $\vec{F}$ about 'O' is defined as $\vec{M} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector of P w.r.t. 'O'. The direction of $\vec{M}$ is along the normal to the plane OPN such that $\vec{r}, \vec{F}$ and $\vec{M}$ form a right handed system.



- Moment of the couple $= (\vec{r}_1 - \vec{r}_2) \times \vec{F}$, where $\vec{r}_1$ and $\vec{r}_2$ are position vectors of the point of the application of the forces $\vec{F}$ and $-\vec{F}$.

DIRECTION COSINES AND DIRECTION RATIOS OF A LINE

- (i) Direction cosine of a line has the same meaning as d.c's of a vector.

  (ii) Any three numbers a, b, c proportional to the direction cosines are called the direction ratios *i.e.*,

  $$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

  same sign either +ve or –ve should be taken through out.
  Note that d.r's of a line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1, y_2 - y_1$ and $z_2 - z_1$.

(iii) If θ is the angle between the two lines whose d.c's are $<l_1, m_1, n_1>$ and $<l_2, m_2, n_2>$, then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

Hence if lines are perpendicular, then $l_1l_2 + m_1m_2 + n_1n_2 = 0$.

If lines are parallel, then $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

Note that if three lines are coplanar,

then $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$

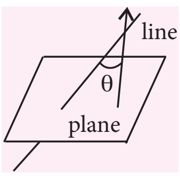
- Projection of the join of two points (x_1, y_1, z_1) and (x_2, y_2, z_2) on a line with d.c's $<l, m, n>$ are $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

PLANE

- General equation of plane is $ax + by + cz + d = 0$.
- Equation of a plane passing through (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.
- Equation of a plane if its intercepts on the coordinate axes are x_1, y_1, z_1 is $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$.
- Equation of a plane if the length of the perpendicular from the origin on the plane is p and d.c's of the perpendicular as $<l, m, n>$ is $lx + my + nz = p$
- **Parallel and perpendicular planes :** Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are
  (i) perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$
  (ii) parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
  (iii) coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$
- Angle between a plane and a line is the complement of the angle between the normal to the plane and the line. If line : $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane : $\vec{r}\cdot\vec{n} = d$, then

$$\cos(90° - \theta) = \sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$$

where θ is the angle between the line andnormal to the plane.



- Length of the perpendicular from a point (x_1, y_1, z_1) to a plane $ax + by + cz + d = 0$ is

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

- Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$\left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$$

- Planes bisecting the angle between two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\left|\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right| = \pm\left|\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

Of these two bisecting planes, one bisects the acute and the other obtuse angle between the given planes.

- Equation of a plane through the intersection of two planes P_1 and P_2 is given by $P_1 + \lambda P_2 = 0$

STRAIGHT LINE IN SPACE

- Equation of a line through $A(x_1, y_1, z_1)$ and having direction cosines $<l, m, n>$ are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

and the equation of a line through (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

- Intersection of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ together represent the unsymmetrical form of the straight line.
- General equation of the plane containing the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$, where $Al + Bm + Cn = 0$.

PROBLEMS

Single Correct Answer Type

1. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and say $\vec{r}$ be a variable vector such that $\vec{r}\cdot\hat{i}, r\cdot\hat{j}, \vec{r}\cdot\hat{k}$ are positive integers. If $\vec{r}\cdot\vec{a}\le 10$, then the number of values of $\vec{r}$ is

(a) ${}^{10}C_3$ (b) ${}^{10}C_3 - 1$
(c) ${}^{10}C_3 + 1$ (d) None of these

2. Let $a, b > 0$ and $\vec{\alpha}=\frac{\hat{i}}{a}+\frac{4\hat{j}}{b}+b\hat{k}$ & $\vec{\beta}=b\hat{i}+a\hat{j}+\frac{1}{b}\hat{k}$, then the maximum value of $\frac{10}{5+\vec{\alpha}\cdot\vec{\beta}}$ is

(a) 1 (b) 2 (c) 4 (d) 8

3. If $\vec{p}, \vec{q}$ are two non-collinear vectors such that $(b-c)(\vec{p}\times\vec{q})+(c-a)\vec{p}+(a-b)\vec{q}=\vec{0}$ where a, b, c are lengths of sides of a triangle, then the triangle is

(a) right angled (b) obtuse angled
(c) equilateral (d) right angled isosceles

4. A line passes through the points (6, –7, –1) and (2, –3, 1). The direction cosines of the line so directed that the angle made by it with the positive direction of x-axis is acute, is

(a) $\frac{2}{3},-\frac{2}{3},-\frac{1}{3}$ (b) $-\frac{2}{3},\frac{2}{3},\frac{1}{3}$
(c) $\frac{2}{3},-\frac{2}{3},\frac{1}{3}$ (d) $\frac{2}{3},\frac{2}{3},\frac{1}{3}$

5. Two adjacent sides of a parallelogram $ABCD$ are given by $\overrightarrow{AB}=2\hat{i}+10\hat{j}+11\hat{k}$ and $\overrightarrow{AD}=-\hat{i}+2\hat{j}+2\hat{k}$. The side AD is rotated by an acute angle α in the plane of parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB then the cosine of angle α is given by

(a) $\frac{8}{9}$ (b) $\frac{\sqrt{17}}{9}$ (c) $\frac{1}{9}$ (d) $\frac{4\sqrt{5}}{9}$

6. If A, B, C and D are four points in space satisfying $AB\cdot CD = k[|AD|^2 + |BC|^2 - |AC|^2 - |BD|^2]$ then the value of k is

(a) 2 (b) 1/3 (c) 1/2 (d) 1

7. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=-\hat{i}+\hat{j}+\hat{k}$, $\vec{c}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{d}=\hat{i}+\hat{j}-\hat{k}$. Then, the line of intersection of planes one determined by $\vec{a}, \vec{b}$ and other determined by $\vec{c}, \vec{d}$ is perpendicular to

(a) x-axis (b) y-axis
(c) both x and y-axes (d) both y and z-axes

8. If $x = cy + bz$, $y = az + cx$, $z = bx + ay$, where x, y, z are not all zero, then $a^2 + b^2 + c^2 =$

(a) $1 + 2abc$ (b) $1 - 2abc$
(c) $1 + abc$ (d) $abc - 1$

9. The equation of the plane through the point (–1, 2, 0) and parallel to the lines $\frac{x}{3}=\frac{y+1}{0}=\frac{z-2}{-1}$ and $\frac{z-2}{-1}=\frac{2y+1}{2}=\frac{z+1}{-1}$ is

(a) $2x + 3y + 6z - 4 = 0$ (b) $x - 2y + 3z + 5 = 0$
(c) $x + y - 3z + 1 = 0$ (d) $x + 2y + 3z - 3 = 0$

10. Let a, b, c be distinct non-negative numbers and the vectors $a\hat{i}+a\hat{j}+c\hat{k}, \hat{i}+\hat{k}, c\hat{i}+c\hat{j}+b\hat{k}$, lie in a plane, then the quadratic equation $ax^2 + 2cx + b = 0$ has

(a) real and equal roots (b) real unequal roots
(c) unreal roots (d) None of these

11. The equation of plane through points (1, 0, –1), (3, 2, 2) and parallel to the line $\frac{x-1}{1}=\frac{1-y}{-2}=\frac{z-2}{3}$ is

(a) $4x - y - 2z - 6 = 0$ (b) $5x + y - 2z - 7 = 0$
(c) $4x + y - 2z - 7 = 0$ (d) $4x - y + 2z + 1 = 0$

12. A unit normal vector to the plane, $x + 2y + 3z - 5 = 0$ is

(a) $\frac{-1}{\sqrt{14}}\hat{i}+\frac{2}{\sqrt{14}}\hat{j}+\frac{3}{\sqrt{14}}\hat{k}$ (b) $\frac{1}{\sqrt{14}}\hat{i}+\frac{2}{\sqrt{14}}\hat{j}+\frac{3}{\sqrt{14}}\hat{k}$
(c) $\frac{1}{\sqrt{14}}\hat{i}+\frac{-2}{\sqrt{14}}\hat{j}+\frac{3}{\sqrt{14}}\hat{k}$ (d) $\frac{1}{\sqrt{14}}\hat{i}+\frac{2}{\sqrt{14}}\hat{j}+\frac{-3}{\sqrt{14}}\hat{k}$

13. If P, Q, R are the images of point $A(a, b, c)$ in XY, YZ and ZX planes respectively and 'G' is the centroid of ΔPQR, then area of ΔAOG = _____ (O refers to origin).

(a) 0 (b) $a^2 + b^2 + c^2$
(c) $\frac{2}{3}(a^2+b^2+c^2)$ (d) $\frac{3}{4}(a^2+b^2+c^2)$

14. The equation of the plane through the intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ and passing through the point (1, 1, 1) is

(a) $20x - 23y + 26z - 69 = 0$
(b) $20x - 23y - 26z - 69 = 0$
(c) $20x - 23y + 26z + 69 = 0$
(d) $20x + 23y + 26z - 69 = 0$

15. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors and $\vec{d}$ be a non-zero vector, which is perpendicular to $\vec{a}+\vec{b}+\vec{c}$. Now, if $\vec{d}=(\sin x)(\vec{a}\times\vec{b})+(\cos y)(\vec{b}\times\vec{c})+2(\vec{c}\times\vec{a})$, then minimum value of $x^2 + y^2$ is equal to

(a) π^2 (b) $\frac{\pi^2}{2}$ (c) $\frac{\pi^2}{4}$ (d) $\frac{5\pi^2}{4}$

16. Angle between the lines $3x + 2y + z - 5 = 0 = x + y - 2z - 3$ and $x - y + 4z = 0 = x + y - 4z$ is equal to

(a) $\cos^{-1}\left(\frac{51}{5\sqrt{29}}\right)$ (b) $\cos^{-1}\left(\frac{29}{5\sqrt{51}}\right)$

(c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\frac{\pi}{2}$

17. Let $A(1, 1, 1)$, $B(2, 3, 5)$, $C(-1, 0, 2)$ be three points then equation of a plane parallel to the plane ABC which is at a distance 3 is

(a) $2x - 3y + z + 3\sqrt{14} = 0$
(b) $2x - 3y + z - 2\sqrt{14} = 0$
(c) $2x - 3y + z + 2 = 0$
(d) $2x - 3y + z + \sqrt{14} = 0$

18. A line passing through $A(1, 2, 3)$ and having direction ratios $(3, 4, 5)$ meets a plane $x + 2y - 3z = 5$ at B, then length of AB is equal to

(a) $\frac{9}{4}$ (b) $\frac{11}{4\sqrt{2}}$ (c) $\frac{13}{4}$ (d) $\frac{45\sqrt{2}}{4}$

19. The distance of the point $P(3, 8, 2)$ from the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$ measured parallel to the plane $3x + 2y - 2z + 15 = 0$ is

(a) 7 (b) 2 (c) 9 (d) 5

20. The volume of the tetrahedron included between the plane $3x + 4y - 5z - 60 = 0$ and the coordinate planes is

(a) 60 (b) 600 (c) 720 (d) 400

21. For unit vectors $\vec{b}$ and $\vec{c}$ and for any non zero vector $\vec{a}$, the value of $\{\{(\vec{a}+\vec{b})+(\vec{a}+\vec{c})\}\times(\vec{b}+\vec{c})\}\cdot(\vec{b}+\vec{c})$ is

(a) $|\vec{a}|^2$ (b) $2|\vec{a}|^2$ (c) $3|\vec{a}|^2$ (d) 0

22. The position vector of a point P is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, where $x, y, z \in N$ and $\vec{a} = \hat{i} + \hat{j} + \hat{k}$. If $\vec{r}\cdot\vec{a} = 12$, then number of possible positions of P is

(a) 45 (b) 55
(c) 35 (d) None of these

23. A variable plane at a distance of the one unit from the origin cuts the coordinates axes at A, B and C. If the centroid $D(x, y, z)$ of triangle ABC satisfies the relation $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$, then find the value of k.

(a) 3 (b) 1 (c) 1/3 (d) 9

24. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is equal to

(a) 1 (b) 4/3 (c) 3/4 (d) 4/5

25. A triangle ABC has vertices $A(1, -2, 2)$, $B(1, 4, 0)$ and $C(-4, 1, 1)$. Find the vector BM, where M is the foot of the altitude drawn from B to AC.

(a) $-\frac{20}{3}\hat{i} - 10\hat{j} + \frac{10}{3}\hat{k}$ (b) $-\frac{10}{7}\hat{i} - \frac{30}{7}\hat{j} + \frac{10}{7}\hat{k}$

(c) $\frac{20}{7}\hat{i} + 5\hat{j} - \frac{10}{7}\hat{k}$ (d) $-\frac{20}{7}\hat{i} - \frac{30}{7}\hat{j} + \frac{10}{7}\hat{k}$

Multiple Correct Answer Type

26. A vector of magnitude 2 along a bisector of the angle between the two vectors $2\hat{i} - 2\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} - 2\hat{k}$ is

(a) $\frac{2}{\sqrt{10}}(3\hat{i} - \hat{k})$ (b) $\frac{1}{\sqrt{26}}(\hat{i} - 4\hat{j} + 3\hat{k})$

(c) $\frac{2}{\sqrt{26}}(\hat{i} - 4\hat{j} + 3\hat{k})$ (d) $\frac{2}{\sqrt{5}}(\hat{i} - 3\hat{j})$

27. If $\vec{a}$ and $\vec{b}$ are any two unit vectors, then possible integer(s) in the range of $\frac{3|\vec{a}+\vec{b}|}{2} + 2|\vec{a}-\vec{b}|$ is

(a) 2 (b) 3 (c) 4 (d) 5

28. A point Q at a distance 3 from the point $P(1, 1, 1)$ lying on the line joining the points $A(0, -1, 3)$ and Q has the coordinates

(a) $(2, 3, -1)$ (b) $(4, 7, -5)$
(c) $(0, -1, 3)$ (d) $(-2, -5, 7)$

29. Let a plane passes through origin and is parallel to the line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z+1}{-2}$ such that distance between plane and the line is 5/3. Then equation of the plane is

(a) $x - 2y + 2z = 0$ (b) $x - 2y - 2z = 0$
(c) $2x + 2y + z = 0$ (d) $x + y + z = 0$

30. Find the ratio in which the join of $(1, -2, 3)$ and $(4, 2, -1)$ is divided by the XOY plane.

(a) 1 : 3 (b) 3 : 1
(c) the point divides the line segment internally
(d) 4 : 1

31. If $\vec{a}$ is perpendicular to $\vec{b}$ and p is a non-zero scalar such that $p\vec{r} + (\vec{r}\cdot\vec{b})\vec{a} = \vec{c}$, then

(a) $[\vec{r}\ \vec{a}\ \vec{c}] = 0$ (b) $p^2\vec{r} = p\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$

(c) $p^2\vec{r} = p\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ (d) $p^2\vec{r} = p\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}$

32. If x and y are two unit vectors and ϕ is the angle between them, then $\frac{1}{2}|x - y|$ is equal to

(a) 0 (b) $\frac{1}{\sqrt{2}}\sqrt{1-\cos\phi}$

(c) $\left|\sin\frac{\phi}{2}\right|$ (d) $\left|\cos\frac{\phi}{2}\right|$

33. If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors, then which of the following statement(s) is/are true ?

(a) $\vec{a}\times(\vec{b}\times\vec{c}), \vec{b}\times(\vec{c}\times\vec{a}), \vec{c}\times(\vec{a}\times\vec{b})$ form a right handed system.

(b) $\vec{c},(\vec{a}\times\vec{b})\times\vec{c}, \vec{a}\times\vec{b}$ form a right handed system.

(c) $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}<0$ if $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

(d) $\dfrac{(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})}{(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{c})}=-1$ if $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

34. The plane $x - 2y + 7z + 21 = 0$

(a) contains the line $\dfrac{x+1}{-3}=\dfrac{y-3}{2}=\dfrac{z+2}{1}$.

(b) contains the point $(0, 7, -1)$.

(c) is perpendicular to the line $\dfrac{x}{1}=\dfrac{y}{-2}=\dfrac{z}{7}$.

(d) is parallel to the plane $x - 2y + 7z = 0$.

35. If the direction ratios of a line are $1 + \lambda, 1 - \lambda, 2$ and it makes an angle 60° with the y-axis then λ is

(a) $1 + \sqrt{3}$ (b) $2 + \sqrt{5}$

(c) $1 - \sqrt{3}$ (d) $2 - \sqrt{5}$

36. Let PM be the perpendicular from the point $P(1, 2, 3)$ to xy plane. If $\overrightarrow{OP}$ makes an angle θ with the positive direction of z-axis and $\overrightarrow{OM}$ makes an angle ϕ with the positive direction of x-axis, where O is the origin and θ and ϕ are acute angles, then

(a) $\tan\theta=\dfrac{\sqrt{5}}{3}$ (b) $\sin\theta\sin\phi=\dfrac{2}{\sqrt{14}}$

(c) $\tan\phi = 2$ (d) $\cos\theta\cos\phi=\dfrac{1}{\sqrt{14}}$

37. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, non-zero vectors and $\vec{r}$ is any vector in space then $[\vec{b}\ \vec{c}\ \vec{r}]\vec{a}+[\vec{c}\ \vec{a}\ \vec{r}]\vec{b}+[\vec{a}\ \vec{b}\ \vec{r}]\vec{c}$ is equal to

(a) $3[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (b) $[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$

(c) $[\vec{b}\ \vec{c}\ \vec{a}]\vec{r}$ (d) $6[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$

38. A vector equally inclined to the vectors $\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-\hat{k}$ in the plane containing them is

(a) $\dfrac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$ (b) $\hat{j}-\hat{k}$

(c) $2\hat{i}$ (d) $\hat{i}$

39. Let $\vec{a}, \vec{b}, \vec{c}$ be non-zero vectors and $|\vec{a}|=1$ and $\vec{r}$ is a non-zero vector such that $\vec{r}\times\vec{a}=\vec{b}$ and $\vec{r}\cdot\vec{c}=1$, then

(a) $\vec{a}\perp\vec{b}$ (b) $\vec{r}\perp\vec{b}$

(c) $\vec{r}\cdot\vec{a}=\dfrac{1-[\vec{a}\ \vec{b}\ \vec{c}]}{\vec{a}\cdot\vec{c}}$ (d) $[\vec{r}\ \vec{a}\ \vec{b}]=0$.

40. Let $4\hat{i}+3\hat{j}$ and $\vec{b}$ be two vectors perpendicular to each other in xy-plane. The vectors $\vec{c}$ in the same plane having projections 1 and 2 along $\vec{a}$ and $\vec{c}$ are

(a) $-\dfrac{2}{3}\hat{i}+\dfrac{11}{2}\hat{j}$ (b) $2\hat{i}-\hat{j}$

(c) $-\dfrac{2}{5}\hat{i}+\dfrac{11}{5}\hat{j}$ (d) $\dfrac{2}{3}\hat{i}+\dfrac{11}{2}\hat{j}$

41. If $OABC$ is a tetrahedron such that $OA^2+BC^2=OB^2+CA^2=OC^2+AB^2$, then

(a) $OA\perp BC$ (b) $OB\perp AC$

(c) $OC\perp AB$ (d) $AB\perp AC$

42. A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. The locus of a point equidistant from origin, A, B and C must be

(a) $ayz + bzx + cxy = 2xyz$

(b) $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=1$ (c) $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=2$

(d) $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=3$

Comprehension Type

Paragraph for Q. No. 43 to 45

Let L_1 and L_2 be the lines whose equations are $\dfrac{x-3}{3}=\dfrac{y-8}{-1}=\dfrac{z-3}{1}$ and $\dfrac{x+3}{-3}=\dfrac{y+7}{2}=\dfrac{z-6}{4}$ respectively. A & B are two points on L_1 and L_2 respectively such that AB is perpendicular to both the lines L_1 and L_2.

43. The shortest distance between the lines L_1 and L_2 is

(a) $\sqrt{30}$ (b) $6\sqrt{10}$ (c) $3\sqrt{30}$ (d) $9\sqrt{10}$

44. Coordinates of the point A are

(a) (1, 8, 2) (b) (3, 8, 3)

(c) (−3, 8, 3) (d) None of these

45. Equation of line of shortest distance is

(a) $\dfrac{x-3}{2}=\dfrac{y-8}{5}=\dfrac{z-3}{-1}$ (b) $\dfrac{x+3}{2}=\dfrac{y+7}{5}=\dfrac{z+6}{1}$

(c) $\dfrac{x}{2}=\dfrac{2y-1}{10}=\dfrac{2z-9}{-1}$ (d) none of these

Paragraph for Q. No. 46 to 48

Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are forming a right handed system, if $\vec{a}\times\vec{b}=\vec{c}$, $\vec{b}\times\vec{c}=\vec{a}$, $\vec{c}\times\vec{a}=\vec{b}$.

46. If $\vec{x}=\vec{a}+\vec{b}-\vec{c}$, $\vec{y}=-\vec{a}+\vec{b}-2\vec{c}$, $\vec{z}=-\vec{a}+2\vec{b}-\vec{c}$, then a unit vector normal to the vector $\vec{x}+\vec{y}$ and $\vec{y}+\vec{z}$ is

(a) $\vec{a}$ (b) $\vec{b}$

(c) $\vec{c}$ (d) None of these

47. If vectors $2\vec{a}-3\vec{b}+4\vec{c}$, $\vec{a}+2\vec{b}-\vec{c}$ and $x\vec{a}-\vec{b}+2\vec{c}$ are coplanar, then x =

(a) 8/5 (b) 5/8
(c) 0 (d) None of these

48. Let $\vec{x}=\vec{a}+\vec{b}$, $\vec{y}=2\vec{a}-\vec{b}$, then the point of intersection of straight lines $\vec{r}\times\vec{x}=\vec{y}\times\vec{x}$ and $\vec{r}\times\vec{y}=\vec{x}\times\vec{y}$ is

(a) $2\vec{b}$ (b) $3\vec{b}$
(c) $3\vec{a}$ (d) None of these

Paragraph for Q. No. 49 to 51

Given a point $P(2, 3, -4)$ and a vector $\vec{b}=2\hat{i}-\hat{j}+2\hat{k}$.

49. Vector equation of a plane passing through the point P perpendicular to the vector $\vec{b}$ is

(a) $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=-7$ (b) $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=7$
(c) $\vec{r}\cdot(2\hat{i}+3\hat{j}-4\hat{k})=-7$ (d) $\vec{r}\cdot(2\hat{i}+3\hat{j}-4\hat{k})=7$

50. Cartesian equation of a plane π passing through the point with position vector $\vec{b}$ and perpendicular to the vector $\overline{OP}$, (O being the origin) is

(a) $2x - y + 2z + 7 = 0$ (b) $2x - y + 2z - 7 = 0$
(c) $2x + 3y - 4z + 7 = 0$ (d) $2x + 3y - 4z - 7 = 0$

51. Sum of the lengths of the intercepts made by the plane π on the coordinate axes is

(a) 14 (b) 91/12 (c) 9/7 (d) 5/7

Matrix – Match Type

52. Match the following:

	Column-I		Column-II
A.	If $\vec{a},\vec{b},\vec{c}$ are unit vectors then the value of $\lvert\vec{a}-\vec{b}\rvert^2+\lvert\vec{b}-\vec{c}\rvert^2+\lvert\vec{c}-\vec{a}\rvert^2$ is at most	p.	$\sqrt{59}$
B.	Let $\vec{v}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{w}=\hat{i}+3\hat{k}$. If $\vec{u}$ is a unit vector then maximum value of the scalar triple product $[\vec{u}\ \vec{v}\ \vec{w}]$ is	q.	9
C.	If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect then value of k is	r.	$\sqrt{10}$
		s.	$\frac{9}{2}$

53. Match the following :

	Column-I		Column-II
A.	The value of α for which the vectors $2\hat{i}-\hat{j}+\hat{k}, \hat{i}+2\hat{j}+\alpha\hat{k}$ and $3\hat{i}-4\hat{j}+5\hat{k}$ are coplanar is	p.	4
B.	The area of a parallelogram having sides $\vec{a}=3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}-3\hat{j}+4\hat{k}$ is	q.	-3
C.	If $\vec{r}\cdot\vec{a}=0, \vec{r}\cdot\vec{b}=0$ and $\vec{r}\cdot\vec{c}=0$ for some non-zero vector $\vec{r}$, then the value of $[\vec{a}\ \vec{b}\ \vec{c}]$ is	r.	$10\sqrt{3}$
D.	The volume of parallelopiped whose sides are given $OA=2\hat{i}-3\hat{j}$, $OB=\hat{i}+\hat{j}-\hat{k}$ and $OC=3\hat{i}-\hat{k}$ is	s.	0

Integer Answer Type

54. If $\vec{a},\vec{b}$ are vectors perpendicular to each other and $|\vec{a}|=2$, $|\vec{b}|=3$, $\vec{c}\times\vec{a}=\vec{b}$, then the least value of $2|\vec{c}-\vec{a}|$ is

55. Let $\vec{c}$ be a unit vector coplanar with $\vec{a}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$ such that $\vec{c}$ is perpendicular to $\vec{a}$. If p be the projection of $\vec{c}$ along $\vec{b}$, then evaluate $\frac{\sqrt{11}}{p}$.

56. If $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$ and $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=\frac{1}{2}$ then the value of $4\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2$ is

57. Let the lines

$$\vec{\gamma}=(7\hat{i}+6\hat{j}+2\hat{k})+s(-3\hat{i}+2\hat{j}+4\hat{k}) \text{ and}$$
$$\vec{\gamma}=(5\hat{i}+3\hat{j}+4\hat{k})+t(2\hat{i}+\hat{j}+3\hat{k})$$

be intersected by a line parallel to the vector $2\hat{i}-2\hat{j}-\hat{k}$ at P, Q respectively. Then $|\overline{PQ}|$ =

58. Let $A(2\hat{i}+3\hat{j}+5\hat{k})$, $B(-\hat{i}+3\hat{j}+2\hat{k})$ and $C(\lambda\hat{i}+5\hat{j}+\mu\hat{k})$ are vertices of a triangle and its median through A is equally inclined to the positive directions of the axes. The value of $2\lambda - \mu$ is equal to

59. If $lx + 13y + mz + n = 0$ is the plane through the intersection of the planes $2x + 3y - z + 1 = 0$ and $x + y - 2z + 3 = 0$ and is perpendicular to the plane $3x - y - 2z = 4$ then $l + m + n$ =

60. If $\vec{a},\vec{b},\vec{c}$ are the three unit vectors and α, β, γ are scalars such that $\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma(\vec{a}\times\vec{b})$. It is given that $\vec{a}\cdot\vec{b}=0$ and $\vec{c}$ makes equal angle with both $\vec{a}$ and $\vec{b}$ then evaluate $\alpha^2+\beta^2+\gamma^2$.

SOLUTIONS

1. (a) : Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then $\vec{r}\cdot\vec{a} \le 10$

$\Rightarrow$ $x + y + z \le 10$

$\therefore$ Number of values of $\vec{r}$

= the number of + ve integral solutions of $x + y + z \le 10 \le$ the sum of the coefficients of exponent of x^{10} in $(x + x^2 + ...\infty)^3$

i.e. the sum of all coefficients of exponent $\le x^{10}$ in $x^3 (1 + x + x^2 + ...\infty)^3$

i.e. the sum of all coefficients of exponent $\le x^7$ in $(1 + x + x^2 + ...\infty)^3$

i.e. (the sum of the constant term + coefficient of x^1 + coefficient of x^2 + ... + coefficient of x^7) in $[(1 - x)^{-1}]^3$

i.e. (constant term *i.e.* coefficient of x^0) + coefficient of x^1 + ... + coefficient of x^7 in $(1 - x)^{-3}$

i.e. (constant term *i.e.* coefficient of x^0) + coefficient of x^1 + ... + coefficient of x^7 in $[1 + {}^3C_1x^1 + {}^4C_2x^2 + {}^5C_3x^3 + {}^6C_4x^4 + {}^7C_5x^5 + {}^8C_6x^6 + {}^9C_7x^7]$

$\therefore$ Required number of values of r

$= ({}^3C_0 + {}^3C_1 + {}^4C_2 + {}^5C_3 + {}^6C_4 + {}^7C_5 + {}^8C_6 + {}^9C_7)$

$= {}^3C_0 + {}^3C_1 + {}^4C_2 + {}^5C_2 + {}^6C_2 + {}^7C_2 + {}^8C_2 + {}^9C_2$

$= {}^4C_1 + {}^4C_2 + ... + {}^9C_2 = {}^{10}C_7 = {}^{10}C_3$

2. (a) : $\vec{\alpha}\cdot\vec{\beta} = \dfrac{b}{a} + \dfrac{4a}{b} + 1$ as $\dfrac{b}{a} + \dfrac{4a}{b} + 1 \ge 5$

So $\left(\dfrac{10}{5 + \vec{\alpha}\cdot\vec{\beta}}\right)_{max} = 1.$

3. (c) : Since, $\vec{p}\times\vec{q}$, $\vec{p}, \vec{q}$ are non-coplanar vectors.

$\Rightarrow$ $b - c = 0, c - a = 0, a - b = 0$

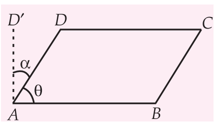
$\Rightarrow$ $a = b = c$ $\Rightarrow$ Triangle is equilateral.

4. (a) : Let l, m, n be the dc's of the given line. Then as it makes an acute angle with x-axis, therefore $l > 0$. The line passes through $(6, -7, -1)$ and $(2, -3, 1)$, therefore its dr's are

$(6 - 2, -7 + 3, -1 - 1)$ or $(4, -4, -2)$ or $(2, -2, -1)$

$\therefore$ Dc's of the given line are $\dfrac{2}{3}, -\dfrac{2}{3}, -\dfrac{1}{3}$.

5. (b): Since, $\cos\theta = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AD}}{|\overrightarrow{AB}||\overrightarrow{AD}|} = \dfrac{8}{9}$



$\therefore$ $\cos^{-1}\left(\dfrac{8}{9}\right) + \alpha = \dfrac{\pi}{2}$ (by hypothesis)

$\therefore$ $\sin\alpha = \dfrac{8}{9}$ $\Rightarrow$ $\cos\alpha = \sqrt{1 - \dfrac{64}{81}} = \dfrac{\sqrt{17}}{9}$

6. (c) : Let A be the origin of reference and let the position vectors of B, C, D be $\vec{b}$, $\vec{c}$, $\vec{d}$. So, $AB = \vec{b}$, $CD = \vec{d} - \vec{c}$, $AD = \vec{d}$, $BC = \vec{c} - \vec{b}$, $AC = \vec{c}$ and $BD = \vec{d} - \vec{b}$.

Then L.H.S. is equal to $\vec{b}\cdot(\vec{d} - \vec{c})$.

R.H.S. $= k[|\vec{d}|^2 + |\vec{c} - \vec{b}|^2 - |\vec{c}|^2 - |\vec{d} - \vec{b}|^2]$.

$= k[\vec{d}\cdot\vec{d} + \vec{c}\cdot\vec{c} + \vec{b}\cdot\vec{b} - 2\vec{c}\cdot\vec{b} - \vec{c}\cdot\vec{c} - \vec{d}\cdot\vec{d} - \vec{b}\cdot\vec{b} + 2\vec{d}.\vec{b}]$

$= 2k[\vec{b}\cdot(\vec{d} - \vec{c})]$

Hence $k = 1/2$

7. (d): $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

$= -4\vec{c} - 4\vec{d} = -8\hat{i}$

Which is perpendicular to both y and z-axes.

8. (b) : $-x + cy + bz = 0$, $cx - y + az = 0$, $bx + ay - z = 0$

$$\Rightarrow \begin{vmatrix} -1 & c & b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0 \Rightarrow a^2 + b^2 + + c^2 = 1 - 2abc$$

9. (d): Let $\vec{a} = 3\hat{i} - \hat{k}$ be the vector parallel to 1st line and $\vec{b} = -\hat{i} + 2\hat{j} - \hat{k}$ be the vector parallel to 2nd line

$$\therefore \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -1 \\ -1 & 2 & -1 \end{vmatrix} = \hat{i}(0+2) - \hat{j}(-3-1) + \hat{k}(6) = 2\hat{i} + 4\hat{j} + 6\hat{k}$$

$\therefore$ Equation of plane is of the form

$2x + 4y + 6z + k = 0$

Since it passes through $(-1, 2, 0)$.

$\therefore$ $-2 + 8 + k = 0$ $\Rightarrow$ $k = -6$

$\therefore$ Equation of plane is $2x + 4y + 6z = 6$

or $x + 2y + 3z = 3$

10. (a) : $\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0 \Rightarrow c^2 - ab = 0$

$\Rightarrow$ $0 = 4c^2 - 4ab \Rightarrow 4(c^2 - ab) = 0$

So, roots are real and equal.

11. (a) : Equation of plane through $(1, 0, -1)$ will be $a(x - 1) + b(y - 0) + c(z + 1)$ passing through $(3, 2, 2)$ gives $2a + 2b + 3c = 0$ and parallel to given line gives $a - 2b + 3c = 0$ $\Rightarrow$ $\dfrac{a}{4} = \dfrac{b}{-1} = \dfrac{c}{-2}$

$\therefore$ Equation of plane will be $4x - y - 2z - 6 = 0$

12. (b) : $x + 2y + 3z - 5 = 0$

$\Rightarrow$ $(x\hat{i} + y\hat{j} + z\hat{k})\cdot(\hat{i} + 2\hat{j} + 3\hat{k}) = 5$

$\Rightarrow$ $\vec{r}\cdot\left(\dfrac{1}{\sqrt{14}}\hat{i} + \dfrac{2}{\sqrt{14}}\hat{j} + \dfrac{3}{\sqrt{14}}\hat{k}\right) = \dfrac{5}{\sqrt{14}}$

$\therefore$ Required unit normal vector is

$$\left(\frac{1}{\sqrt{14}}\hat{i}+\frac{2}{\sqrt{14}}\hat{j}+\frac{3}{\sqrt{14}}\hat{k}\right)$$

13. (a) : $P \equiv (a, b, -c)$; $Q \equiv (-a, b, c)$; $R \equiv (a, -b, c)$

$$\Rightarrow\ G=\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$$

$\therefore$ OAG are collinear.

14. (d) : The equation of any plane through the intersection of the given planes is

$(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0$ ...(i)

If (i) passes through (1, 1, 1), we have

$-3 + 14\lambda = 0$ or $\lambda = \frac{3}{14}$

Putting $\lambda = \frac{3}{14}$ in (i) we get the required equation of the plane as $(x + y + z - 6) + \frac{3}{14}(2x + 3y + 4z + 5) = 0$

or $20x + 23y + 26z - 69 = 0$

15. (d) : Since $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar $\Rightarrow [\vec{a}, \vec{b}, \vec{c}] \neq 0$

Also, $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ are non-coplanar.

Given $\vec{d}=(\sin x)(\vec{a}\times\vec{b})+(\cos y)(\vec{b}\times\vec{c})+2(\vec{c}\times\vec{a})$.

Taking dot product with $\vec{a}+\vec{b}+\vec{c}$, we get

$0=\sin x[\vec{a}\ \vec{b}\ \vec{c}]+\cos y[\vec{a}\ \vec{b}\ \vec{c}]+2[\vec{a}\ \vec{b}\ \vec{c}]$

$\Rightarrow \sin x + \cos y + 2 = 0 \Rightarrow \sin x + \cos y = -2$

$\Rightarrow x=(4n-1)\frac{\pi}{2};\ y=(2n+1)\pi, n\in Z$

For least value of $x^2 + y^2$, $x=\frac{-\pi}{2}, y=\pi$

and least value is $\frac{5\pi^2}{4}$.

16. (b) : Direction ratios of line I is

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 2 & 1\\ 1 & 1 & -2\end{vmatrix}=\hat{i}(-5)-\hat{j}(-7)+\hat{k}(1)=(-5, 7, 1)$$

Direction ratios of line II is

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 4\\ 1 & 1 & -4\end{vmatrix}=\hat{i}(0)-\hat{j}(-8)+\hat{k}(2)=(0, 8, 2)$$

Let θ be angle between the lines, then

$$\cos\theta=\frac{(-5\times 0)+(7\times 8)+(1\times 2)}{\sqrt{25+49+1}\sqrt{0+64+4}}$$

$$\Rightarrow\ \theta=\cos^{-1}\left\{\frac{58}{(5\sqrt{3})(2\sqrt{17})}\right\}=\cos^{-1}\left\{\frac{29}{5\sqrt{51}}\right\}$$

17. (a) : Dr's of AB are 1, 2, 4; Dr's of AC are $-2, -1, 1$

Dr's of normal to plane ABC are 2, -3, 1

$\therefore$ Equation of the plane ABC is $2x - 3y + z = 0$

Let the equation of required plane be

$2x - 3y + z = k$, then

$$\left|\frac{2\times 1-3\times 1+1-k}{\sqrt{4+9+1}}\right|=3\ \Rightarrow\ k=\pm 3\sqrt{14}$$

(since distance of point (1, 1, 1) from required plane = 3)

Then, equation of required plane is,

$2x - 3y + z \mp 3\sqrt{14} = 0$

18. (d) : The line through $A(1, 2, 3)$, having dr's (3, 4, 5) is $\vec{r}=(1, 2, 3)+t(3, 4, 5)$

Any point lies on it will be $(1 + 3t, 2 + 4t, 3 + 5t)$, t being scalar.

If $(1 + 3t, 2 + 4t, 3 + 5t)$ lies on the plane, then

$1 + 3t + 2(2 + 4t) - 3(3 + 5t) = 5$

$$\Rightarrow\ -4t=9\ \Rightarrow\ t=\frac{-9}{4}$$

So the point is $\left(\frac{-23}{4}, \frac{-28}{4}, \frac{-33}{4}\right)=B$

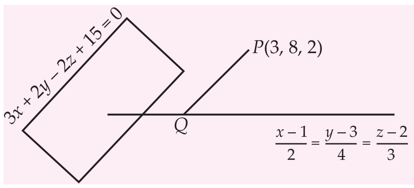
$$\therefore\ \text{Length of } AB=\sqrt{\left(1+\frac{23}{4}\right)^2+\left(2+\frac{28}{4}\right)^2+\left(3+\frac{33}{4}\right)^2}$$

$$=\frac{1}{4}\sqrt{27^2+36^2+45^2}=\frac{45\sqrt{2}}{4}$$

19. (a) : Here, $Q=(2r+1, 4r+3, 3r+2)$

Dr's of $PQ = (2r - 2, 4r - 5, 3r)$

Dr's of normal to plane = $(3, 2, -2)$



PQ is perpendicular to plane

$3(2r - 2) + 2(4r - 5) - 2(3r) = 0 \Rightarrow r = 2$

$\therefore$ $Q = (5, 11, 8)$.

$\therefore$ Required distance = $PQ = 7$

20. (b) : Equation of the given plane can be written as $\frac{x}{20}+\frac{y}{15}+\frac{z}{-12}=1$ which meets the coordinates axes in points $A(20, 0, 0)$, $B(0, 15, 0)$ and $C(0, 0, -12)$ and the coordinates of the origin are (0, 0, 0).

$\therefore$ Volume of the tetrahedron $OABC$ is

$$\left|\frac{1}{6}\begin{vmatrix} 20 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & -12 \end{vmatrix}\right| = \left|\frac{1}{6}\times 20\times 15\times(-12)\right| = 600$$

21. (d)

22. (b) : Given, $\vec{r}\cdot\vec{a} = 12$

$\Rightarrow$ $x + y + z = 12$ and $x, y, z \in N$

$\therefore$ Required number of positions of P

= The number of positive solutions of the equations

$x + y + z = 12$

= The coefficient of x^{12} in $(x + x^2 + x^3 + ...)^3$

= Coefficient of x^{12} in $x^3\,(1 + x + x^2 + ...)^3$

= Coefficient of x^9 in$((1 - x)^{-1})^3$

= Coefficient of x^9 in $(1 - x)^{-3}$

$= (1 + {}^3C_1x + {}^4C_2x^2 + ... + {}^{11}C_9x^9 + ...)$

$= {}^{9+3-1}C_{3-1} = {}^{11}C_2 = 55$

23. (d) : Let the equation of variable plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ which meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Centroid of ΔABC is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ and it satisfied the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=k \Rightarrow \frac{9}{a^2}+\frac{9}{b^2}+\frac{9}{c^2}=k$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{k}{9} \qquad ...(i)$$

Also given that the distance of plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ from(0, 0, 0) is 1 unit.

$$\Rightarrow \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=1 \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=1 \qquad ...(ii)$$

From (i) and (ii) we get $k = 9$

24. (b) :

25. (d) : Suppose M divides AC in the ratio l : 1. Then,

$$M=\left(\frac{-4\lambda+1}{\lambda+1}, \frac{\lambda-2}{\lambda+1}, \frac{\lambda+2}{\lambda+1}\right)$$

Hence $\overrightarrow{BM}=\left(\frac{-5\lambda}{\lambda+1}, \frac{-3\lambda-6}{\lambda+1}, \frac{\lambda+2}{\lambda+1}\right)$

and $\overrightarrow{AC}=(-5,3,-1)$.

Now $\overrightarrow{BM}\perp\overrightarrow{AC}$

$\Rightarrow$ $25\lambda - 9\lambda - 18 - \lambda - 2 = 0 \Rightarrow \lambda = \frac{4}{3}$

$\therefore$ $\overrightarrow{BM}=\left(\frac{-20}{7}, \frac{-30}{7}, \frac{10}{7}\right)$

26. (a, c) : $\vec{a}=2\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

$\therefore$ $|\vec{a}| = 3$, $|\vec{b}| = 3$

$\therefore$ A vector along bisectors

$$=\frac{\vec{a}}{|\vec{a}|}\pm\frac{\vec{b}}{|\vec{b}|}=\hat{i}-\frac{1}{3}\hat{k}, \frac{1}{3}\hat{i}-\frac{4}{3}\hat{j}+\hat{k}$$

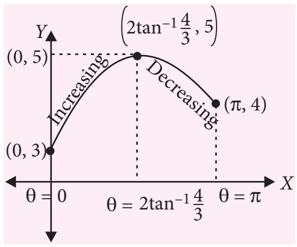
$\therefore$ The required vector

$$=2.\frac{\hat{i}-\frac{1}{3}\hat{k}}{\sqrt{1^2+\left(-\frac{1}{3}\right)^2}}, 2.\frac{\frac{1}{3}\hat{i}-\frac{4}{3}\hat{j}+\hat{k}}{\sqrt{\left(\frac{1}{3}\right)^2+\left(-\frac{4}{3}\right)^2+1^2}}$$

$$=\frac{2}{\sqrt{10}}(3\hat{i}-\hat{k}), \frac{2}{\sqrt{26}}(\hat{i}-4\hat{j}+3\hat{k})$$

27. (b, c, d) : Let angle between $\vec{a}$ and $\vec{b}$ be θ.

We have, $|\vec{a}|=|\vec{b}|=1$



Now, $|\vec{a}+\vec{b}|=2\cos\frac{\theta}{2}$

and $|\vec{a}-\vec{b}|=2\sin\frac{\theta}{2}$

Consider $F(\theta)=\frac{3}{2}\left(2\cos\frac{\theta}{2}\right)+2\left(2\sin\frac{\theta}{2}\right)$

$\therefore$ $F(\theta)=3\cos\frac{\theta}{2}+4\sin\frac{\theta}{2}, \theta\in[0,\pi]$

$\therefore$ $F'(\theta)=\frac{-3}{2}\sin\frac{\theta}{2}+2\cos\frac{\theta}{2}$

Now, $F'(\theta) = 0$ $\Rightarrow$ $\tan\frac{\theta}{2}=\frac{4}{3}$

Clearly, $F(0) = 3$

$$F\left(2\tan^{-1}\frac{4}{3}\right)=3\left(\frac{3}{5}\right)+4\left(\frac{4}{5}\right)=5$$

$F(\pi) = 4$ $\therefore$ Range = [3, 5]

Hence possible integer(s) in the range of $F(\theta)$ in $[0, \pi]$ are three.

28. (a, c) : If Q is (α, β, γ) then (α, β, γ), (1, 1, 1) and (0, –1, 3) are collinear.

$$\therefore \frac{\alpha-1}{1-0}=\frac{\beta-1}{1-(-1)}=\frac{\gamma-1}{1-3}.$$

Also, $PQ = 3$

$\Rightarrow$ $(\alpha - 1)^2 + (\beta - 1)^2 + (\gamma - 1)^2 = 9$

Solve these equations for α, β, γ, we get

$Q \equiv (2, 3, -1), (0, -1, 3)$

29. (a, c) : Let the equation of plane be

$lx + my + nz = 0$, where l, m, n be d.c's

$\Rightarrow \quad l^2 + m^2 + n^2 = 1 \qquad \text{...(i)}$

Given line :

$$\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z+1}{-2}$$

$\therefore \quad 2l - m - 2n = 0 \qquad \text{...(ii)}$

Also $\dfrac{l-3m-n}{\sqrt{l^2+m^2+n^2}} = \dfrac{5}{3} \Rightarrow l - 3m - n = \dfrac{5}{3} \qquad \text{...(iii)}$

Solving (i), (ii) and (iii) we get equation of plane as

$x - 2y + 2z = 0$ or $2x + 2y + z = 0$.

30. (b, c) : Let $A(1, -2, 3)$ and $B(4, 2, -1)$. Let the plane XOY meet the line AB in the point C such that C divides $[AB]$ in the ratio $k : 1$, then

$$C \equiv \left(\frac{4k+1}{k+1}, \frac{2k-2}{k+1}, \frac{-k+3}{k+1}\right).$$

Since C lies on the plane XOY *i.e.*, the plane $z = 0$, therefore,

$$\frac{-k+3}{k+1} = 0 \Rightarrow k = 3$$

31. (a, d) : Let $\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a} \times \vec{b})$.

Taking dot product with $\vec{b}$

$\Rightarrow \vec{r} \cdot \vec{b} = 0 + y|\vec{b}|^2 \Rightarrow \vec{a}(\vec{r} \cdot \vec{b}) = y\vec{b}^2\vec{a}$

$\Rightarrow \vec{c} - p\vec{r} = y|\vec{b}|^2\,\vec{a} \Rightarrow \vec{r} = \dfrac{1}{p}\vec{c} - \dfrac{y|\vec{b}|^2}{p}\vec{a}$

$\therefore \ [\vec{r}\ \vec{a}\ \vec{c}] = 0.$

Now $\vec{r} \cdot \vec{b} = \dfrac{1}{p}\vec{c} \cdot \vec{b}$

$\therefore \ y|\vec{b}|^2 = \dfrac{\vec{b} \cdot \vec{c}}{p} \qquad \therefore \ \vec{r} = \dfrac{1}{p}\vec{c} - \dfrac{1}{p^2}(\vec{b} \cdot \vec{c})\vec{a}$

32. (b, c) : $|x - y|^2 = |x|^2 + |y|^2 - 2xy = 1 + 1 - 2\cos\phi$

$= 2(1 - \cos\phi)$

$\Rightarrow \quad |x - y| = \sqrt{2}\sqrt{1-\cos\phi}$

$\Rightarrow \quad \dfrac{1}{2}|x-y| = \dfrac{1}{\sqrt{2}}\sqrt{1-\cos\phi} = \left|\sin\dfrac{\phi}{2}\right|$

33. (b, c, d) :

(a) $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$

$\Rightarrow$ vectors are coplanar, so do not form RHS.

(b) $(\vec{a} \times \vec{b}) \times \vec{c},\ \vec{a} \times \vec{b},\ \vec{c}$ in that order form RHS

$\Rightarrow \ \vec{c}, (\vec{a} \times \vec{b}) \times \vec{c}, \vec{a} \times \vec{b}$ also form RHS as they are in same cyclic order.

(c) $\vec{a} + \vec{b} + \vec{c} = \vec{0} \Rightarrow (\vec{a} + \vec{b} + \vec{c})^2 = 0$

$\Rightarrow \ \vec{a}^2 + \vec{b}^2 + \vec{c}^2 = -2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$

Hence $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} < 0$

(d) $\vec{a} + \vec{b} + \vec{c} = \vec{0} \Rightarrow \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Using this we get result.

34. (a, b, c, d) : Conditions for the plane $ax + by + cz + d = 0$ to contain the line $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ are $a\alpha + b\beta + c\gamma + d = 0$ and $al + bm + cn = 0$.

Here the condition is satisfied for option (a). For option (c), normal to plane and given line have same d.r's.

$\therefore$ The line is perpendicular to the given plane.

For option (b), clearly the point satisfies the given plane.

The d.r's of normal to the plane in option (d) are $(1, -2, 7)$ which are same as the d.r's of normal to the given plane.

$\therefore$ The plane given in option (d) is parallel to given plane.

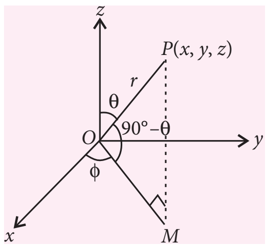
35. (b, d) : $\dfrac{1-\lambda}{\sqrt{(1+\lambda)^2 + (1-\lambda)^2 + 4}} = \cos 60°$

$\Rightarrow \quad 4(1 - \lambda)^2 = 2(1 + \lambda^2) + 4$

$\Rightarrow \quad \lambda^2 - 4\lambda - 1 = 0$

$\Rightarrow \quad \lambda = \dfrac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5} \quad \therefore \quad \lambda = 2 + \sqrt{5}$ or $2 - \sqrt{5}$.

36. (a, b, c) : If P be (x, y, z) then from the figure,



$OM = r\cos(90° - \theta)$

$= r\sin\theta$

$\Rightarrow \quad x = r\sin\theta\cos\phi,$

$y = r\sin\theta\sin\phi,\ z = r\cos\theta$

$\Rightarrow \quad 1 = r\sin\theta\cos\phi,$

$2 = r\sin\theta\sin\phi,\ 3 = r\cos\theta$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 = r^2 \Rightarrow r = \pm\sqrt{14}$

$\therefore \ \sin\theta\cos\phi = \dfrac{1}{\sqrt{14}},\ \sin\theta\sin\phi = \dfrac{2}{\sqrt{14}},\ \cos\theta = \dfrac{3}{\sqrt{14}}$

(Neglecting negative sign as θ and ϕ are acute)

$\therefore \quad \dfrac{\sin\theta\sin\phi}{\sin\theta\cos\phi} = \dfrac{2}{1} \Rightarrow \tan\phi = 2.$

Also, $\tan\theta = \dfrac{\sqrt{5}}{3}.$

37. (b, c)

38. (c, d) : A vector coplanar with the given vectors is $(1+\lambda)\hat{i}+(\lambda-1)\hat{j}+(1-\lambda)\hat{k}$

Since it is equally inclined to the two given vectors

$$\therefore \quad \frac{(1+\lambda)\hat{i}+(\lambda-1)\hat{j}+(1-\lambda)\hat{k}}{|(1+\lambda)\hat{i}+(\lambda-1)\hat{j}+(1-\lambda)\hat{k}|}\cdot\frac{(\hat{i}-\hat{j}+\hat{k})}{\sqrt{3}}$$

$$= \frac{(1+\lambda)\hat{i}+(\lambda-1)\hat{j}+(1-\lambda)\hat{k}}{|(1+\lambda)\hat{i}+(\lambda-1)\hat{j}+(1-\lambda)\hat{k}|}\cdot\frac{(\hat{i}+\hat{j}-\hat{k})}{\sqrt{3}}$$

$\therefore \quad \lambda = 1$

$\Rightarrow$ Required vector is $2\hat{i}$ or $\hat{i}$.

39. (a, b, c) : $(\vec{a}\times\vec{c})\cdot(\vec{r}\times\vec{a})=(\vec{a}\times\vec{c})\cdot\vec{b}$

40. (b, c) : Let $\vec{b}=x\hat{i}+y\hat{j}$. Since $\vec{a}$ is perpendicular to $\vec{b}$ so $4x + 3y = 0$. Thus $\vec{b} = x\left(\hat{i}-\frac{4}{3}\hat{j}\right)$. Let $\vec{c}=u\hat{i}+v\hat{j}$ be the required vector. According to the given condition

$1 = \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|} \Rightarrow 4u + 3v = \pm 5$. Also $2 = \frac{\vec{c}\cdot\vec{b}}{|\vec{b}|}$

$$\Rightarrow \quad \frac{ux-(4/3)vx}{\sqrt{x^2(1+(16/9))}} = 2 \Rightarrow 3u - 4v = \pm 10$$

Solving these equations we have $u = 2$ and $v = -1$ or $u = -2/5$, $v = 11/5$

$$\therefore \quad \vec{c}=(2\hat{i}-\hat{j})\left(-\frac{2}{5}\hat{i}+\frac{11}{5}\hat{j}\right)$$

41. (a, b, c) : Let $\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=\vec{b}, \overrightarrow{OC}=\vec{c}$, then

$\vec{a}\cdot\vec{a}+(\vec{b}-\vec{c})\cdot(\vec{b}-\vec{c})=\vec{b}\cdot\vec{b}+(\vec{c}-\vec{a})\cdot(\vec{c}-\vec{a})$

$\Rightarrow \quad -2\vec{b}\cdot\vec{c}=-2\vec{c}\cdot\vec{a}\Rightarrow(\vec{a}-\vec{b})\cdot\vec{c}=0$

or $\overrightarrow{BA}\cdot\overrightarrow{OC}=0$. Hence $\overrightarrow{AB}\perp\overrightarrow{OC}$

Similarly, $\overrightarrow{BC}\perp\overrightarrow{OA}$ and $\overrightarrow{CA}\perp\overrightarrow{OB}$.

42. (a,c): Let A, B, C be $(\alpha, 0, 0)$, $(0, \beta, 0)$ and $(0, 0, \gamma)$ then the plane ABC is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$

Since it always passes through a, b, c. So,

$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$ ...(i)

If P is (u, v, w) then $OP^2 = AP^2 = BP^2 = CP^2$

$\Rightarrow \quad u^2 + v^2 + w^2 = (u - \alpha)^2 + v^2 + w^2$

$\Rightarrow \quad \alpha = 2u, \beta = 2v, \gamma = 2w$

On putting α, β, γ in (i) we get $\frac{a}{u}+\frac{b}{v}+\frac{c}{w}=2$

$\Rightarrow$ Locus of (u, v, w) is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

43. (b) : Let $\vec{a}=(3,8,3), \vec{c}=(-3,-7,6)$,

$\vec{b}=3\hat{i}-\hat{j}+\hat{k},\ \vec{d}=-3\hat{i}+2\hat{j}+4\hat{k}$

Shortest distance between L_1, L_2 is

$$\left|\frac{(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})}{|\vec{b}\times\vec{d}|}\right| = \frac{270}{|-6\hat{i}-15\hat{j}+3\hat{k}|} = \frac{270}{\sqrt{270}} = \sqrt{270}$$

44. (b) : Notice that line joining the points (3, 8, 3), (–3, –7, 6) is line with d.r's 6, 15, –3 & is perpendicular to both L_1 & L_2.

So take A = (3, 8, 3), B = (–3, –7, 6)

45. (a) : Line of shortest distance is line joining AB only in this case (given in passage)

As d.r's of AB are 6, 15, –3 or 2, 5, –1

$\therefore \quad \overleftrightarrow{AB}$ is $\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$

46. (d) : $\vec{x}+\vec{y}=2\vec{b}-3\vec{c}$ and $\vec{y}+\vec{z}=-2\vec{a}+3\vec{b}-3\vec{c}$

$$\therefore \quad (\vec{x}+\vec{y})\times(\vec{y}+\vec{z})=\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ 0 & 2 & -3\\ -2 & 3 & -3\end{vmatrix}=3\vec{a}+6\vec{b}+4\vec{c}$$

$\therefore$ Required unit vector $= \frac{3\vec{a}+6\vec{b}+4\vec{c}}{\sqrt{61}}$

47. (a) : $\begin{vmatrix}2 & -3 & 4\\ 1 & 2 & -1\\ x & -1 & 2\end{vmatrix}=0$

$\Rightarrow \quad 2(4-1)+3(2+x)+4(-1-2x)=0 \Rightarrow x=\frac{8}{5}$

48. (c) :

$\vec{r}\times\vec{x}=\vec{y}\times\vec{x} \Rightarrow (\vec{r}-\vec{y})\times\vec{x}=0 \Rightarrow \vec{r}=\vec{y}+\lambda\vec{x}$

$\vec{r}\times\vec{y}=\vec{x}\times\vec{y} \Rightarrow (\vec{r}-\vec{x})\times\vec{y}=0 \Rightarrow \vec{r}=\vec{x}+\mu\vec{y}$

$\Rightarrow \quad \vec{y}+\lambda\vec{x}=\vec{x}+\mu\vec{y}$

$\Rightarrow \quad (2\vec{a}-\vec{b})+\lambda(\vec{a}+\vec{b})=(\vec{a}+\vec{b})+\mu(2\vec{a}-\vec{b})$

$\Rightarrow \quad 2 + \lambda = 1 + 2\mu, -1 + \lambda = 1 - \mu \Rightarrow \mu = 1, \lambda = 1$

$\therefore$ The point of intersection is $3\vec{a}$.

49. (a) : Equation of the plane through P whose position vector is $\vec{a}=2\hat{i}+3\hat{j}-4\hat{k}$, perpendicular to $\vec{b}$ is $(\vec{r}-\vec{a})\cdot\vec{b}=0$.

$\Rightarrow \quad \vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=(2\hat{i}+3\hat{j}-4\hat{k})\cdot(2\hat{i}-\hat{j}+2\hat{k})$

$= 2 \times 2 + 3(-1) - 4 \times 2 = -7$

50. (c) : Equation of the plane p is $2x + 3y - 4z = -7$

51. (b) : Sum of the intercepts of the plane p on the coordinate axis $= \left|-\frac{7}{2}\right|+\left|\frac{-7}{3}\right|+\left|\frac{-7}{-4}\right|=\frac{91}{12}$

52. A → q ; B → p ; C → s

A. Let $x=\Sigma|\vec{a}-\vec{b}|^2=6-2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

Also $|\vec{a}+\vec{b}+\vec{c}|^2\geq 0 \Rightarrow \Sigma\vec{a}\cdot\vec{b}\geq\frac{-3}{2}$

$\therefore \quad$ Max(x) = 9

B. $|\vec{u}|=1, \vec{u}\cdot\vec{v}\times\vec{w}=\sqrt{59}\cos\theta$

C. $\therefore$ Since, the given lines intersect,

$(2r+1, 3r-1, 4r+1)=(s+3, 2s+k, s)$

$\Rightarrow\ r=\dfrac{-3}{2}, s=-5 \Rightarrow k=\dfrac{9}{2}$

53. A → q; B → r; C → s; D → p

A. Since, vectors are coplanar

$\therefore \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & \alpha \\ 3 & -4 & 5 \end{vmatrix}=0 \Rightarrow \alpha=-3$

B. Area of parallelogram $=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}=10\sqrt{3}$

C. Since, $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\therefore [\vec{a}\ \vec{b}\ \vec{c}]=0$

D. Volume of parallelopiped $=\begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}=4$

54. (3) : $\vec{c}\times\vec{a}=\vec{b} \Rightarrow |\vec{c}\times\vec{a}|=|\vec{b}| \Rightarrow |\vec{c}||\vec{a}|\sin\theta=3,$

$\Rightarrow\ |\vec{c}|=\dfrac{3}{2\sin\theta}$

$\Rightarrow |\vec{c}-\vec{a}|^2=|\vec{c}|^2+|\vec{a}|^2-2\vec{c}\cdot\vec{a}$

$=|\vec{c}|^2+4-2|\vec{c}|\cdot|\vec{a}|\cos\theta$

$=\dfrac{9}{4\sin^2\theta}+4-2\cdot\dfrac{3}{2\sin\theta}\cdot 2\cdot\cos\theta$

$=4+\dfrac{9}{4}\text{cosec}^2\theta-6\cot\theta=\dfrac{9}{4}+\left(\dfrac{3}{2}\cot\theta-2\right)^2$

$\Rightarrow\ |\vec{c}-\vec{a}|^2\geq\dfrac{9}{4} \Rightarrow |\vec{c}-\vec{a}|\geq\dfrac{3}{2}$

$\Rightarrow\ 2|\vec{c}-\vec{a}|\geq 3 \therefore$ Min. of $2|\vec{c}-\vec{a}|=3$

55. (6) : Let $\vec{c}=x\vec{a}+y\vec{b}$, where x, y are scalars

$\Rightarrow\ \vec{c}=x(\hat{i}-\hat{j}+2\hat{k})+y(2\hat{i}-\hat{j}+\hat{k})$

$\Rightarrow\ \vec{c}=\hat{i}(x+2y)+\hat{j}(-x-y)+\hat{k}(2x+y)$

But $\vec{c}\cdot\vec{a}=0 \Rightarrow 6x+5y=0 \Rightarrow y=\dfrac{-6x}{5}$

So, $\vec{c}=\dfrac{-7x}{5}\hat{i}+\dfrac{x}{5}\hat{j}+\dfrac{4x}{5}\hat{k}$

We have $\dfrac{49x^2+x^2+16x^2}{25}=1 \Rightarrow x^2=\dfrac{25}{66}$

$\therefore\ \vec{c}=\pm\dfrac{5}{\sqrt{66}}\left(\dfrac{-7}{5}\hat{i}+\dfrac{1}{5}\hat{j}+\dfrac{4}{5}\hat{k}\right);\ p=\dfrac{|\vec{c}\cdot\vec{b}|}{|\vec{b}|}=\dfrac{\sqrt{11}}{6}$

So, $\dfrac{\sqrt{11}}{p}=6$

56. (2) : $[\vec{a}\ \vec{b}\ \vec{c}]^2=[\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}]$

$=\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}=\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix}=\dfrac{1}{2}$

$\Rightarrow\ 4[\vec{a}\ \vec{b}\ \vec{c}]^2=2$

57. (9) : Let $P\equiv(7-3s, 6+2s, 2+4s)$

and $Q\equiv(5+2t, 3+t, 4+3t)$

Since, PQ is parallel to $\vec{r}=(2\hat{i}-2\hat{j}-\hat{k})$

i.e. $\overrightarrow{PQ}=\lambda(2\hat{i}-2\hat{j}-\hat{k})$

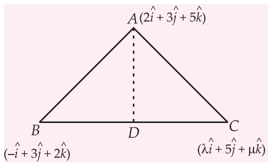
$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=(-2+2t+3s, -3+t-2s, 2+3t-4s)$

$\therefore\ -2+2t+3s=2\lambda;\ -3+t-2s=-2\lambda;$

$2+3t-4s=-\lambda$

Solving, we get $\lambda=3$, $\therefore\ |\overrightarrow{PQ}|=3|2\hat{i}-2\hat{j}-\hat{k}|=9$

58. (4) : P.V. of D $=\dfrac{\lambda-1}{2}\hat{i}+4\hat{j}+\dfrac{\mu+2}{2}\hat{k}$



D.r.'s of AD are $\dfrac{\lambda-5}{2}, 1, \dfrac{\mu-8}{2}$.

But direction cosines of AD should be $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

$\Rightarrow\ \dfrac{\lambda-5}{2}=1=\dfrac{\mu-8}{2} \Rightarrow \lambda=7, \mu=10$

$\therefore\ 2\lambda-\mu=4$

59. (2) : The given plane will be of the form $2x+3y-z+1+\lambda(x+y-2z+3)=0$. If this is perpendicular to $3x-y-2z-4=0$, then

$3(\lambda+2)+(-1)(3+\lambda)+(-2)(-1-2\lambda)=0$

$\Rightarrow\ 6\lambda+5=0 \Rightarrow \lambda=-5/6$

$\therefore$ The plane is $7x+13y+4z-9=0$

$\Rightarrow\ l+m+n=7+4-9=2$

60. (1) : Let $\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}=\cos\theta$ and also $\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma(\vec{a}\times\vec{b})$

Taking dot product with $\vec{a}$ both sides, $\cos\theta=\alpha$

Taking dot product with $\vec{b}$ both sides, $\cos\theta=\beta$

Taking dot product with $\vec{c}$ both sides

$1=\alpha\cos\theta+\beta\cos\theta+\gamma[\vec{a}\ \vec{b}\ \vec{c}]$

But $[\vec{a}\ \vec{b}\ \vec{c}]^2=\begin{vmatrix} 1 & 0 & \cos\theta \\ 0 & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}=1-2\cos^2\theta$

So, $1=\cos^2\theta+\cos^2\theta+\gamma\sqrt{1-2\cos^2\theta}$

$\Rightarrow \gamma=\sqrt{1-2\cos^2\theta}$

So, $\alpha^2+\beta^2+\gamma^2=1$

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CLASS XII **Series 9**

ACE YOUR WAY CBSE

Linear Programming | Probability

HIGHLIGHTS

LINEAR PROGRAMMING

Some Definitions

Optimisation Problems
A problem in which a function is to be optimised (maximised/minimised) satisfying certain linear inequalities.

Decision Variables
The variables that enter into the problem.

LPP
A problem of finding optimal (maximum or minimum) value of a linear function subject to certain restrictions (constraints) determined by a set of linear inequations with variable as non-negative.

Objective Function
The linear function which is to be optimized is called a objective function.

Linear Constraints
The linear inequations (inequalities) or restrictions on the decision variables of a linear programming problem.

Convex Region
A region or a set of points in which the line joining any two of its points lies completely in the region.

Feasible Region
The common region determined by all the constraints including non-negative constraints of a LPP.

Corner Points
Corner points of a feasible region are points of intersection of any two boundary lines.

Feasible Solution
A feasible solution is the set of values of the variables satisfying all the linear constraints and non-negativity restrictions.

Optimal Solution
A feasible solution of a LPP which optimizes (maximizes or minimizes) the objective function.

MATHEMATICAL FORMULATION OF L.P.P.

The procedure for mathematical formulation of a linear programming problem consists of the following steps :

Step I In every L.P.P. certain decisions are to be made. These decisions are represented by decision variables. These decision variables are those quantities whose values are to be determined. Identify the variables and denote them by $x_1, x_2, x_3, \ldots$..

Previous Years Analysis						
	2016		2015		2014	
	Delhi	AI	Delhi	AI	Delhi	AI
VSA	–	–	–	–	–	–
SA	1	1	1	1	1	1
LA	2	2	2	2	2	2

Step II Identify the objective function and express it as a linear function of the variables introduced in Step I.

Step III In a L.P.P., the objective function may be in the form of maximising profits or minimising costs. So, after expressing the objective functions as a linear function of the decision variables, we must find the type of optimisation *i.e.,* maximisation or minimisation. Identify the type of the objective function.

Step IV Identify the set of the constraints, stated in terms of decision variables and express them as linear inequations or equations as the case may be.

IMPORTANT THEOREMS

- Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at the corner point (vertex) of the feasible region.
- Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.

  Note : If R is unbounded, then a maximum or minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R.

GRAPHICAL METHODS OF SOLVING L.P.P.

The graphical methods for solving L.P.P. is applicable to those problems which involve only two variables.

Corner Point Method

To solve a linear programming problem by corner point method, we follow the following steps :

(i) Formulate the given L.P.P. in mathematical form.

(ii) Convert all inequations into equations and draw their graphs.

(iii) Determine the region represented by each inequation.

(iv) Obtain the region in xy-plane containing all points that simultaneously satisfy all constraints including non-negativity restrictions. The region so obtained is the feasible region.

(v) Determine the coordinates of the vertices (corner points) of the feasible region obtained in Step (iv). These vertices are known as the extreme points of the set of all feasible solutions of the L.P.P.

(vi) Obtain the values of the objective function at each of the vertices and let M and m are the maximum and minimum values respectively at the vertices.

(vii) If feasible region is bounded then M and m are the maximum and minimum values of objective function.

(viii) If feasible region is unbounded then

- M is Maximum value of $Z = ax + by$ if open half plane determined by $ax + by > M$ has no point in common with feasible region otherwise no maximum value.
- m is maximum value of $Z = ax + by$ if open half plane determined by $ax + by < m$ has no point in common with feasible region otherwise no minimum value.

DIFFERENT TYPES OF L.P.P.

Following are the important types of linear programming problems:

- **Diet problems :** In diet problems, amounts of different kinds of nutrients/constituents which are to be included in a diet are determined so as to minimize the cost of the desired diet such that it contains a certain minimum amount of each nutrient/constituent.
- **Manufacturing problems (Optimal product line problems) :** In such type of problems, the number of units of different products which is to be produced and sold by a firm when each product requires a fixed man power, machine hour, labour hour per unit of product, ware house space per unit of the output etc., are to be determined so as to achieve maximum profit.
- **Transportation problems :** In such type of problems, a transportation schedule so as to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets are to be determined so as to minimize the cost of transportation subject to limitations (constraints) of demand of each market and supply from each plant or factory.

PROBABILITY

CONDITIONAL PROBABILITY

If two events A and B are associated with the same random experiment, the conditional probability $P(A|B)$ of the occurrence of A knowing that B has occurred is given by

$$P(A\,|\,B)=\frac{P(A\cap B)}{P(B)}, P(B)\neq 0$$

Similarly, $P(B\,|\,A)=\dfrac{P(A\cap B)}{P(A)}, P(A)\neq 0\cdot$

Properties of Conditional Probability

- If E_1 and E_2 are any two events associated with an experiment and $P(E_2) \neq 0$, then $0 \leq P(E_1|E_2) \leq 1$.
- If E is any event associated with an experiment, then $P(E|E) = P(S|E) = 1$, S being the sample space.
- If E, E_1 and E_2 are events associated with an experiment and $P(E) \neq 0$, then
  $P((E_1 \cup E_2)|E) = P(E_1|E) + P(E_2|E) - P((E_1 \cap E_2)|E)$
  In particular, if E_1 and E_2 are mutually exclusive, then
  $P((E_1 \cup E_2)|E) = P(E_1|E) + P(E_2|E)$
- If E_1 and E_2 are any two events associated with an experiment, then $P(E_1^c|E_2) = 1 - P(E_1|E_2)$

MULTIPLICATION THEOREM ON PROBABILITY

If A and B are two events associated with a random experiment, then

$$P(A \cap B) = P(A) \cdot P(B|A)\ ;\ P(A) \neq 0$$

or $$P(A \cap B) = P(B) \cdot P(A|B)\ ;\ P(B) \neq 0$$

If $A_1, A_2, A_3, ..., A_n$ are n events related to a random experiment, then

$$P(A_1 \cap A_2 \cap .... \cap A_n) = P(A_1) \times P(A_2|A_1) \times P(A_3|(A_1 \cap A_2)) \times .... \times P(A_n|(A_1 \cap A_2 \cap ... \cap A_{n-1})),$$

where $P(A_n|(A_1 \cap A_2 \cap ..... \cap A_{n-1}))$ represents the conditional probability of the event A_n, given that the events $A_1, A_2, ..., A_{n-1}$ have already happened.

INDEPENDENT EVENTS

Two random experiments are said to be independent iff the probability of occurrence or non-occurrence of any event E_2 associated with the second experiment is independent of the outcome of the first experiment. This means that if E_1 is any event associated with the first experiment, then

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2)$$

Note :

- Two events E and F are said to be dependent if they are not independent, *i.e.*, if
  $P(E \cap F) \neq P(E) \cdot P(F)$
- Three events A, B and C are said to be mutually independent, if
  $$P(A \cap B) = P(A) \cdot P(B)$$
  $$P(A \cap C) = P(A) \cdot P(C)$$
  $$P(B \cap C) = P(B) \cdot P(C)$$
  and $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

  If at least one of the above is not true for three given events, we say that the events are not independent.

THEOREM OF TOTAL PROBABILITY

Partition of a sample space : A set of events $E_1, E_2, ..., E_n$ is said to represent a partition of the sample space S if

(i) $E_i \cap E_j = \phi$, $i \neq j$, $i, j = 1, 2, 3 ....., n$

(ii) $E_1 \cup E_2 \cup ..... \cup E_n = S$

(iii) $P(E_i) > 0$ for all $i = 1, 2, ..., n$.

For example any non empty event E and its complement E' form a partition of the sample space S since they satisfy $E \cap E' = \phi$ and $E \cup E' = S$.

Law of total probability : Let S be the sample space and let $E_1, E_2, ..., E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event of non-zero probability which occurs with E_1 or E_2 or ..... or E_n, then

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + ... + P(E_n)P(A|E_n)$$

$$=\sum_{i=1}^{n} P(E_i)P(A\,|\,E_i)$$

BAYES' THEOREM

Let S be the sample space and let $E_1, E_2, ..., E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event of non- zero probability which occurs with E_1 or E_2 or ..... or E_n, then

$$P(E_i\,|\,A)=\frac{P(E_i)P(A\,|\,E_i)}{\sum_{i=1}^{n} P(E_i)P(A\,|\,E_i)}, i=1,2, ..., n$$

RANDOM VARIABLE AND ITS PROBABILITY DISTRIBUTION

- **Random variable :** Random variable is simply a variable whose values are determined by the outcomes of a random experiment; generally it is denoted by capital letters such as X, Y, Z etc. and their values are denoted by the corresponding small letters x, y, z etc.

- **Probability distribution :** The system consisting of a random variable X along with $P(X)$, is called the probability distribution of X.
  The probability distribution of a random variable X is the system of numbers

X	:	x_1	x_2	.....	x_n
$P(X)$	:	p_1	p_2	.....	p_n

where $p_i > 0, \sum_{i=1}^{n} p_i = 1, i = 1, 2, ..., n$

- **Mean of a Random Variable**
  The mean of a random variable X is also called the expectation of X, denoted by μ or $E(X)$.
  So, $E(X) = \mu = p_1x_1 + p_2x_2 + ... + p_nx_n = \sum_{i=1}^{n} p_ix_i$

- **Variance of a Random Variable**

$$\text{Var}(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p_i$$

$$\text{or Var}(X) = \sum_{i=1}^{n} p_ix_i^2 - \left(\sum_{i=1}^{n} p_ix_i\right)^2 = E(X^2) - [E(X)]^2$$

$$\sigma_x = \sqrt{\text{Var}(X)} = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 p_i}$$

is called the Standard Deviation of the random variable X.

BERNOULLI TRIALS AND BINOMIAL DISTRIBUTION

- **Bernoulli trials :** A sequence of independent trials which can result in one of the two mutually exclusive possibilities success or failure such that the probability of success or failure in each trial is constant, then such repeated independent trials are called Bernoulli trials.
- **Binomial distribution :** A random variable X taking values 0, 1, 2, ..., n is said to have a binomial distribution with parameters n and p, if its probability distribution is given by

$$P(X = r) = {}^nC_r p^r q^{n-r}$$

where p represents probability of success and q represents probability of failure and n is the number of trials.
Note : All the conditions of the Bernoulli trials should be satisfied while using above probability density functions of the binomial distribution in solving any problem.

- **Mean, variance and standard deviation :**
  (i) Mean $=np$
  (ii) Variance $= npq$
  (iii) Standard deviation $= \sqrt{npq}$

PROBLEMS

Very Short Answer Type

1. If $P(A) = \frac{3}{8}, P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$, then find $P(A'|B')$.
2. A die is tossed thrice. Getting an even number is considered as success. What is the variance of the binomial distribution?
3. A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event' 3 on the die'. Check whether A and B are independent events or not.
4. Find the expectation of the number of heads in 15 tosses of a coin.
5. The probability distribution of X is :

X	0	1	2	3
$P(X)$	0.2	K	K	$2K$

Find the value of K.

Short Answer Type

6. Evaluate $P(A \cup B)$, if $2P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$.
7. Two dice are thrown. Find the probability of getting an odd number on the first and a multiple of 3 on the other.
8. The probability distribution of a random variable X is given below:

X :	0	1	2	3	4
$P(X)$:	0.1	0.25	0.3	0.2	0.15

Find Var(X).

9. A bag contains 3 white and 6 black balls while another bag contains 6 white and 3 black balls. A bag is selected at random and a ball is drawn. Find the probability that the ball drawn is of white colour.
10. If the probability of defective bolts is 0.1, find the mean and standard deviation for the distribution of defective bolts in a total of 500 bolts.

Long Answer Type-I

11. The postmaster of a local post office wishes to hire extra helpers during the deepawali season, because of a large increase in the volume of mail handling and delivery. Because of the limited office space, and the budgetary condition, the number of temporary helpers must not exceed 10. According to the past experience, men can handle 300 letters and 80 packages per day on the average and women can handle 400 letters and 50 packages per day. The postmaster believes that the daily volume of the extra mail and packages will be no less than 3400 and 680 respectively. A man receives ₹ 25 a day and woman receives ₹ 22 a day, formulate the problem as a linear programming problem to minimise the payroll.

12. Find the probability of at most two tails or at least two heads in a toss of three coins.

13. Solve the following problem graphically :

Minimize and Maximize : $Z = 3x + 9y$

Subject to constraints : $x + 3y \le 60$, $x + y \ge 10$, $x \le y$, $x \ge 0$, $y \ge 0$

14. The probabilities of P, Q and R solving a problem are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. If the problem is attempted by all simultaneously, find the probability of exactly one of them solve it.

15. A shopkeeper deals in two items-wall hangings and artificial plants. He has ₹ 15000 to invest and a space to store atmost 80 pieces. A wall hanging costs him ₹ 300 and an artificial plant ₹ 150. He can sell a wall hanging at a profit of ₹ 50 and an artificial plant at a profit of ₹ 18. Assuming that he can sell all the items that he buys, formulate a linear programming problem in order to maximize his profit.

Long Answer Type-II

16. A brick manufacturer has two depots, A and B, with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three builders P, Q and R for 15,000, 20,000 and 15,000 bricks respectively. The cost (in ₹) transporting 1000 bricks to the builders from the depots are given below :

From \ To	P	Q	R
A	40	20	30
B	20	60	40

How should the manufacturer fulfill the orders so as to keep the cost of transportation minimum?

17. The sum of mean and variance of a binomial distribution is 15 and the sum of their squares is 117. Determine the distribution.

18. Two persons A and B throw a coin alternately till one of them gets 'head' and wins the game. Find their respective probabilities of winning.

19. There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on two cards drawn. Find the mean and variance of X.

20. In a bolt factory, machines A, B and C manufacture 25%, 35% and 40% respectively. Of the total of their output 5%, 4% and 2% are defective. A bolt is drawn and is found to be defective. What are the probabilities that it was manufactured by the machines A, B and C?

SOLUTIONS

1. $P(A'|B') = \frac{P(A' \cap B')}{P(B')} = \frac{P\{(A \cup B)'\}}{P(B')} = \frac{1-\frac{3}{4}}{1-\frac{5}{8}} = \frac{2}{3}$

2. Here, $n = 3$

p = Probability of getting an even number in a single throw of a die.

$\Rightarrow \quad p = \frac{3}{6} = \frac{1}{2}$ and $q = 1 - p = \frac{1}{2}$

$\therefore$ Variance $= npq = 3 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$

3. We have, $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{6}$

Also, $P(A \cap B) = P$ (Head appears on coin and 3 on the die) $= \frac{1}{12}$

Clearly, $P(A \cap B) = P(A) \times P(B)$

Thus, A and B are independent events.

4. Let p be the probability of getting a head in a single toss. Then, $p = \frac{1}{2}$.

Clearly, the distribution of the number of heads is a binomial distribution with $n = 15$ and $p = \frac{1}{2}$.

$\therefore$ Expectation $= E(X) = np = 15 \times \frac{1}{2} = 7.5$

5. We have,

$\Sigma P(X) = 1$

$\Rightarrow$ $0.2 + 4K = 1 \Rightarrow 4K = 0.8 \Rightarrow K = 0.2$

6. We have, $2P(A) = P(B) = \frac{5}{13}, P(A \mid B) = \frac{2}{5}$.

Therefore $P(A) = \frac{5}{26}$

Since, $P(A \cap B) = P(A|B) \cdot P(B) = \frac{2}{13}$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{11}{26}$$

7. Let A = the event of getting an odd number on first die

and B = the event of getting a multiple of 3 on the second die.

Then, $A = \{1, 3, 5\}$ and $B = \{3, 6\}$

$\therefore$ $P(A) = \frac{3}{6} = \frac{1}{2}$ and $P(B) = \frac{2}{6} = \frac{1}{3}$

Now, required probability $= P(A \cap B)$

$= P(A) \cdot P(B)$ [$\because$ A and B are independent]

$= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

8. Mean of $X = E(X) = \sum_{i=1}^{n} x_i p_i$

$= 0 \times 0.1 + 1 \times 0.25 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.15$
$= 2.05$

and $E(X^2) = 0^2 \times 0.1 + 1^2 \times 0.25 + 2^2 \times 0.3 + 3^2 \times 0.2 + 4^2 \times 0.15 = 5.65$

Now, $\text{Var}(X) = E(X^2) - [E(X)]^2$
$= 5.65 - (2.05)^2 = 1.4475$

9.

Bag I	**Bag II**
3 W	6 W
6 B	3 B

Let E_1 = event that bag I is selected

E_2 = event that bag II is selected

E = event that the ball drawn is of white colour

By rule of total probability,

$P(E) = P(E_1) \cdot P(E|E_1) + P(E_2) \cdot P(E|E_2)$

$$= \frac{1}{2} \cdot \frac{3}{9} + \frac{1}{2} \cdot \frac{6}{9} = \frac{9}{18} = \frac{1}{2}$$

10. We have,

$n = 500$ and $p = 0.1$, $q = 1 - p = 0.9$

$\therefore$ Mean $= np = 500 \times 0.1 = 50$

And,

$S.D. = \sqrt{\text{Variance}} = \sqrt{npq} = \sqrt{500 \times 0.1 \times 0.9} = 6.71$

11. Let x men and y women helpers be hired to minimise the payroll. Therefore, the total payroll per day in rupees, say, Z is given by

$Z = 25x + 22y$

Since the number of helper should not exceed 10.

$\therefore$ $x + y \le 10$

Letters delivered by x men $= 300\,x$

and letters delivered by y women $= 400\,y$

$\therefore$ Total letters delivered by x men and y women $= 300x + 400y$.

Letters expected per day are atleast 3400. Therefore,

$300x + 400y \ge 3400$.

Similarly, for packages, $80x + 50y \ge 680$.

Moreover, x and y, being the number of persons cannot be negative, *i.e.*, $x \ge 0$ and $y \ge 0$.

Therefore, the required linear programming problem is

Minimise $Z = 25x + 22y$

Subject to the constraints :

$x + y \le 10$, $300x + 400y \ge 3400$,

$80x + 50y \ge 680$ and $x, y \ge 0$

12. On tossing three coins, sample space

$S = [HHH, HHT, HTH, THH, TTH, THT, HTT, TTT] \Rightarrow n(S) = 8$

Let A be the event of getting at most two tails, then

$A = [HHH, HHT, HTH, THH, TTH, THT, HTT]$

$\Rightarrow n(A) = 7$

Therefore $P(A) = \frac{n(A)}{n(S)} = \frac{7}{8}$

Let B be the event of getting at least two heads.

Then, $B = [HHT, HTH, THH, HHH] \Rightarrow n(B) = 4$

Therefore, $P(B) = \frac{n(B)}{n(S)} = \frac{4}{8}$

$A \cap B = \{HHH, HHT, HTH, THH\}$

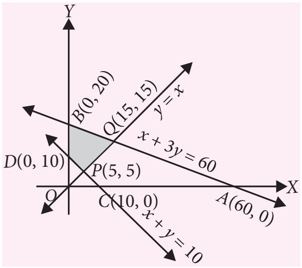
$\Rightarrow$ $P(A \cap B) = \frac{4}{8}$

$\therefore$ Required probability $= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$$= \frac{7}{8} + \frac{4}{8} - \frac{4}{8} = \frac{7}{8}.$$

13. First we draw the lines : $x + 3y = 60$, $x + y = 10$ and $x = y$



The feasible region (shown shaded) is the bounded region $PQBD$.

The vertices of the feasible region are $P(5, 5)$, $Q(15, 15)$, $B(0, 20)$ and $D(0, 10)$

Given $Z = 3x + 9y$

At $D(0, 10)$, $Z = 90$

At $P(5, 5)$, $Z = 60$

At $Q(15, 15)$, $Z = 180$

At $B(0, 20)$, $Z = 180$

Maximum value of $Z = 180$ which occurs at two points $Q(15, 15)$ and $B(0, 20)$.

And minimum value of $Z = 60$ which occurs at $P(5, 5)$.

14. Let A, B, C denote the events that the problem is solved by P, Q and R respectively.

Given $P(A)=\frac{1}{2} \Rightarrow P(A')=1-\frac{1}{2}=\frac{1}{2}$

$P(B)=\frac{1}{3} \Rightarrow P(B')=1-\frac{1}{3}=\frac{2}{3}$

$P(C)=\frac{1}{4} \Rightarrow P(C')=1-\frac{1}{4}=\frac{3}{4}.$

Required probability

$= P[(A\cap B'\cap C')\cup(A'\cap B\cap C')\cup(A'\cap B'\cap C)]$

$= P(A\cap B'\cap C')+P(A'\cap B\cap C')+P(A'\cap B'\cap C)$

$= P(A)\,P(B')\,P(C')+P(A')\,P(B)\,P(C')+P(A')\,P(B')\,P(C)$

[$\because$ A, B, C are independent events]

$=\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}+\frac{1}{2}\times\frac{1}{3}\times\frac{3}{4}+\frac{1}{2}\times\frac{2}{3}\times\frac{1}{4}=\frac{11}{24}.$

15. Let x be the number of wall hangings and y be the number of artificial plants that the dealer buys and sells.

Then the profit of the dealer $= 50x + 18y$

$\therefore$ The objective function is $Z = 50x + 18y$

As a wall hanging costs ₹ 300 and an artificial plant costs ₹ 150, the cost of x wall hangings and y artificial plants is $300x + 150y$. We have given that the dealer can invest atmost ₹ 15000.

$\therefore$ $300x + 150y \le 15000$ (Investment constraint)

or $2x + y \le 100$

As the dealer has space to store atmost 80 pieces, we have another constraint

$x + y \le 80$ (Space constraint)

Also the number of wall hangings and artificial plants can't be negative.

$\therefore$ $x \ge 0, y \ge 0$ (Non-negativity constraints)

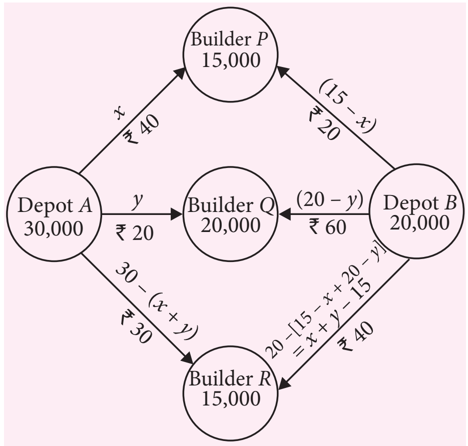
Hence the mathematical formulation of the LPP is

Maximize $Z = 50x + 18y$

Subject to the constraints :

$2x + y \le 100$, $x + y \le 80$, $x \ge 0$, $y \ge 0$

16. Let the depot A transport x thousand bricks to builder P and y thousand bricks to builder Q.



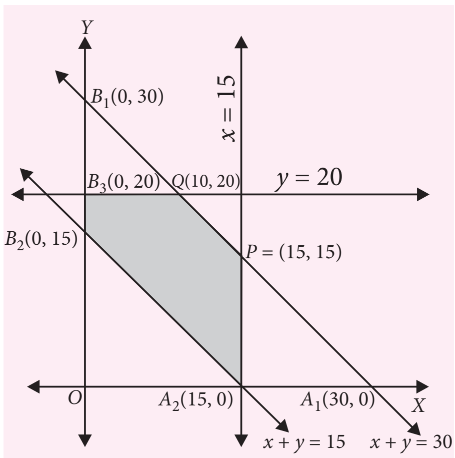
Then, the given LPP can be stated mathematically as follows:

Minimize $Z = 30x - 30y + 1800$

Subject to constraints : $x + y \le 30$, $x \le 15$,

$y \le 20$, $x + y \ge 15$ and $x \ge 0$, $y \ge 0$

To solve this LPP graphically, we first convert inequations into equations and then draw the corresponding lines. The feasible region of the LPP is shaded in the figure. The coordinates of the corner points of the feasible region $A_2PQB_3B_2$ are $A_2(15, 0)$, $P(15, 15)$, $Q(10, 20)$, $B_3(0, 20)$ and $B_2(0, 15)$. These points have been obtained by solving the corresponding intersecting lines simultaneously.



The values of the objective function at the corner points of the feasible region are given in the following table :

Point (x, y)	Value of the objective function $Z = 30x - 30y + 1800$
$A_2(15, 0)$	$Z = 30 \times 15 - 30 \times 0 + 1800 = 2250$
$P(15, 15)$	$Z = 30 \times 15 - 30 \times 15 + 1800 = 1800$
$Q(10, 20)$	$Z = 30 \times 10 - 30 \times 20 + 1800 = 1500$
$B_3(0, 20)$	$Z = 30 \times 0 - 30 \times 20 + 1800 = 1200$
$B_2(0, 15)$	$Z = 30 \times 0 - 30 \times 15 + 1800 = 1350$

Clearly, Z is minimum at $x = 0$, $y = 20$ and the minimum value of Z is 1200.

Thus, the manufacturer should supply 0, 20 and 10 thousand bricks to builders P, Q and R from depot A and 15, 0 and 5 thousand bricks to builders P, Q and R from depot B respectively. In this case the minimum transportation cost will be ₹ 1200.

17. Let n and p be the parameters of the distribution.

Then, Mean = np and Variance = npq.

It is given that

Mean + Variance = 15

and (Mean)^2 + (Variance)^2 = 117

Now, $np + npq = 15$ and $n^2p^2 + n^2p^2q^2 = 117$

$\Rightarrow$ $np(1 + q) = 15$ and $n^2p^2(1 + q^2) = 117$

$\Rightarrow$ $n^2p^2(1 + q)^2 = 225$ and $n^2p^2(1 + q^2) = 117$

$$\Rightarrow \frac{n^2p^2(1+q)^2}{n^2p^2(1+q^2)} = \frac{225}{117} \Rightarrow \frac{(1+q)^2}{(1+q^2)} = \frac{225}{117}$$

$$\Rightarrow \frac{1+q^2+2q}{1+q^2} = \frac{225}{117} \Rightarrow 1+\frac{2q}{1+q^2} = \frac{225}{117}$$

$$\Rightarrow \frac{2q}{1+q^2} = \frac{12}{13} \Rightarrow \frac{1+q^2}{2q} = \frac{13}{12}$$

$$\Rightarrow \frac{1+q^2+2q}{1+q^2-2q} = \frac{13+12}{13-12}$$

[Applying componendo and dividendo]

$$\Rightarrow \left(\frac{1+q}{1-q}\right)^2 = 25 \Rightarrow \frac{1+q}{1-q} = 5 \Rightarrow 6q = 4$$

$$\Rightarrow q = \frac{2}{3}$$

Now, $p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$

Putting $p = \frac{1}{3}, q = \frac{2}{3}$ in $np + npq = 15$, we get

$$\frac{n}{3} + \frac{2n}{9} = 15 \Rightarrow \frac{5n}{9} = 15 \Rightarrow n = 27$$

Thus, $n = 27$, $p = \frac{1}{3}$ and $q = \frac{2}{3}$

Hence, the distribution is given by

$$P(X = r) = {}^{27}C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{27-r}, \; r = 0, 1, 2, \ldots\ldots, 27.$$

18. Let E = the event of occurrence of head in one throw of a coin.

Then, E' = the event of occurrence of tail in one throw of a coin.

$\therefore$ $P(E) = \frac{1}{2}$ and $P(E') = 1 - P(E) = \frac{1}{2}$

Let E_1 = the event that A wins

and E_2 = the event that B wins.

Clearly exactly one of E_1 and E_2 will certainly happen.

$\therefore$ $E_2 = E'_1$

For A to win, he must throw a head in first, third, fifth, ...throws and B should not throw a head in second, fourth, sixth, ... throws.

$$\therefore \; P(E_1) = \frac{1}{2} + \left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\cdot\frac{1}{2} + \left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right) \times \left(1-\frac{1}{2}\right)\frac{1}{2} + \ldots\ldots \text{to } \infty$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \ldots\text{.to } \infty$$

$$= \frac{1}{2}\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \ldots\text{.to } \infty\right]$$

$$= \frac{1}{2} \cdot \frac{1}{\left[1-\left(\frac{1}{2}\right)^2\right]} = \left(\frac{1}{2} \times \frac{4}{3}\right) = \frac{2}{3}$$

and $P(E_2) = P(E'_1) = 1 - P(E_1) = \frac{1}{3}$

Thus, $P(E_1) = \frac{2}{3}$ and $P(E_2) = \frac{1}{3}$

19. The sum X of the numbers on two cards drawn without replacement can take values 3, 4, 5, 6, 7, 8, 9.

The sum can be three if one of the cards drawn bears number 1 and other bears number 2.

$$P(X=3)=P((1,2),(2,1))=\frac{1}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{4}=\frac{2}{20}$$

Similarly,

$$P(X=4)=P((1,3),(3,1))=\frac{1}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{4}=\frac{2}{20}$$

$P(X = 5) = P((1, 4), (4, 1), (2, 3), (3, 2))$

$$=\frac{1}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{4}=\frac{4}{20}$$

$$P(X=6)=P((1,5),(5,1),(2,4),(4,2))=\frac{4}{20}$$

$$P(X=7)=P((2,5),(5,2),(3,4),(4,3))=\frac{4}{20}$$

$$P(X=8)=P((3,5),(5,3))=\frac{2}{20}$$

$$P(X=9)=P((4,5),(5,4))=\frac{2}{20}$$

Thus, the probability distribution of X is as given below.

$X:$	3	4	5	6	7	8	9
$P(X):$	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{4}{20}$	$\frac{4}{20}$	$\frac{2}{20}$	$\frac{2}{20}$

x_i	p_i	$p_i x_i$	$p_i x_i^2$
3	$\frac{2}{20}$	$\frac{6}{20}$	$\frac{18}{20}$
4	$\frac{2}{20}$	$\frac{8}{20}$	$\frac{32}{20}$
5	$\frac{4}{20}$	$\frac{20}{20}$	$\frac{100}{20}$
6	$\frac{4}{20}$	$\frac{24}{20}$	$\frac{144}{20}$
7	$\frac{4}{20}$	$\frac{28}{20}$	$\frac{196}{20}$
8	$\frac{2}{20}$	$\frac{16}{20}$	$\frac{128}{20}$
9	$\frac{2}{20}$	$\frac{18}{20}$	$\frac{162}{20}$
		$\Sigma p_i x_i=\frac{120}{20}=6$	$\Sigma p_i x_i^2=\frac{780}{20}=39$

$\therefore$ Mean $= \Sigma p_i x_i = 6$ and

Variance $=\Sigma p_i x_i^2-(\Sigma p_i x_i)^2=39-36=3$

20. Let E be the event that a bolt drawn is defective.

Let E_1, E_2, E_3 denote the events that the bolt drawn was manufactured by machines A, B and C respectively.

Now $P(E_1)=\frac{25}{100}, P(E_2)=\frac{35}{100}, P(E_3)=\frac{40}{100}$

$P(E|E_1)$ = Probability that one bolt drawn is defective when it has been manufactured by machine A.

$=\frac{5}{100}$

Similarly, $P(E|E_2)=\frac{4}{100}, P(E|E_3)=\frac{2}{100}$

Now $P(E_1|E)$ = Probability that the defective drawn bolt has been manufactured by machine A

$$=\frac{P(E_1)\cdot P(E|E_1)}{P(E_1)\cdot P(E|E_1)+P(E_2)\cdot P(E|E_2)+P(E_3)\cdot P(E|E_3)}$$

[By Bayes' theorem]

$$=\frac{\frac{25}{100}\cdot\frac{5}{100}}{\frac{25}{100}\cdot\frac{5}{100}+\frac{35}{100}\cdot\frac{4}{100}+\frac{40}{100}\cdot\frac{2}{100}}$$

$$=\frac{125}{125+140+80}=\frac{125}{345}=\frac{25}{69}$$

$$P(E_2|E)=\frac{P(E_2)\cdot P(E/E_2)}{P(E_1)\cdot P(E|E_1)+P(E_2)\cdot P(E|E_2)+P(E_3)\cdot P(E|E_3)}$$

$$=\frac{\frac{35}{100}\cdot\frac{4}{100}}{\frac{25}{100}\cdot\frac{5}{100}+\frac{35}{100}\cdot\frac{4}{100}+\frac{40}{100}\cdot\frac{2}{100}}=\frac{140}{345}=\frac{28}{69}$$

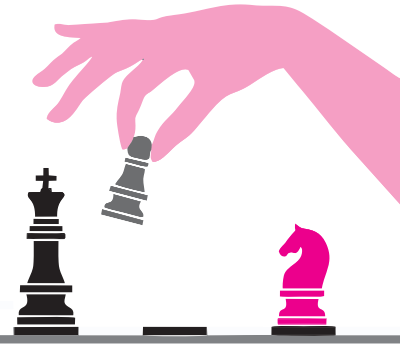
$$P(E_3|E)=1-P(E_1|E)-P(E_2|E)=1-\frac{25}{69}-\frac{28}{69}=\frac{16}{69}$$

◈◈

MPP-7 CLASS XI ANSWER KEY

1. (c)	**2.** (a)	**3.** (b)	**4.** (a)	**5.** (c)
6. (b)	**7.** (b,d)	**8.** (a,b,c,d)	**9.** (a,b,c)	**10.** (a,b,c,d)
11. (a,b)	**12.** (a,b,c,d)	**13.** (a,b)	**14.** (d)	**15.** (b)
16. (c)	**17.** (5)	**18.** (0)	**19.** (7)	**20.** (9)

Challenging PROBLEMS ON Probability



1. The probability that a randomly chosen divisor of 30^{39} is a multiple of 30^{29} is
(a) $\left(\frac{11}{40}\right)^3$ (b) $\left(\frac{21}{40}\right)^3$ (c) $\left(\frac{11}{40}\right)^5$ (d) $\left(\frac{21}{40}\right)^5$

2. 40 slips are placed in a box, each bearing a number 1 through 10 with each number entered on 4 slips. 4 slips are drawn from the box at random without replacement. The probability that two of the slips bear a number 'a' and the other two bear a number 'b' $(\neq a)$ is $k\times\frac{{}^{10}C_2}{{}^{40}C_4}$ where $k=$
(a) 6 (b) 8 (c) 36 (d) 64

3. 10 points in the plane are given, with no 3 points are collinear. 4 distinct segments joining pairs of these points are chosen at random, all such segments being equally likely. The probability that some 3 of the segments form a triangle whose vertices are among the 10 given points is $\frac{k}{{}^{45}C_4}$ where $k=$
(a) 2400 (b) 2040 (c) 5400 (d) 5040

4. Consider the set of lattice points (x, y), $(x, y \in I)$ and $x, y \in [0, 7]$. Two points are selected at random from this set. All points have same probability of being selected and the points need not be distinct. The probability that the area of the triangle formed by these two points and the point $(0, 0)$ is an integer (possibly zero) is
(a) 5/8 (b) 6/8 (c) 7/8 (d) 8/9

5. Let x be chosen at random from the interval $(0, 1)$. The probability that $[\log_{10}4x] = [\log_{10}x]$ is _____ ($[\cdot]$ denotes the greatest integer function)
(a) 1/2 (b) 1/3 (c) 1/6 (d) 5/6

6. Four persons are playing a game of toy-gun shooting. At midnight, each person randomly chooses one of the other three and shoots him. The probability that exactly 2 persons are shot is
(a) $\frac{8}{27}$ (b) $\frac{4}{9}$ (c) $\frac{2}{3}$ (d) $\frac{1}{4}$

7. Let E, F, G be pairwise independent events with $P(G) > 0$ and $P(E \cap F \cap G) = 0$ then
$P(\bar{E}\cap F/\bar{G})+\frac{P(E\cap F)}{P(\bar{G})}=$
(a) $P(E)$ (b) $P(F)$ (c) $P(G)$ (d) $P(E \cup G)$

8. 8 children are standing in a line outside a ticket counter at zoo. 4 of them have a 1-rupee coin each and the remaining four have a 2-rupee coin each. The entry ticket costs 1-rupee each. If all the arrangements of the 8 children are random, the probability that no child will have to wait for a change, if the cashier at the ticket window has no-change to start with is
(a) 1/2 (b) 1/3 (c) 1/4 (d) 1/5

9. 20 chairs are set in a row. 5 people randomly sit on the chairs. The probability that no body is sitting next to anybody else is $\frac{{}^{m}C_5}{{}^{n}C_5}$ where $m + n =$
(a) 16 (b) 20 (c) 36 (d) 26

10. In a certain lottery, 7 balls are drawn at random from n balls numbered 1 through n. If the probability that no pair of consecutive numbers is drawn equals the probability of drawing exactly one pair of consecutive numbers then $n =$
(a) 50 (b) 54 (c) 60 (d) 64

11. If a needle of length 1 unit is dropped at random on a surface ruled with parallel lines at a distance 2 units apart, the probability that the needle will cross one of the lines is
(a) $1/\pi$ (b) $1/2\pi$ (c) $1/3\pi$ (d) $1/4\pi$

12. In a test, each of the 3 candidates receives a problem sheet with n problems from geometry and calculus. The 3 problem sheets contain respectively one, two and three calculus questions. The candidates choose randomly a problem from the sheet and answer it.

By : Tapas Kr. Yogi, Mob : 9533632105.

The probability that all candidates answer geometry problems is

(a) $\frac{n(n-1)(n-2)}{(n-3)^3}$ (b) $\frac{(n-1)(n-2)(n-3)}{n^3}$

(c) $\frac{n(n-1)(n-2)}{(n+1)^2}$ (d) $\frac{(n-3)}{(n-1)(n-2)}$

13. The fraction of all permutations of the numbers 1, 2, 3, 4, 5, 6 such that the first term is not 1 has third term 3 is

(a) $\frac{4}{23}$ (b) $\frac{3}{23}$ (c) $\frac{4}{25}$ (d) $\frac{3}{25}$

14. Two real numbers a and b are chosen at random between 0 and 1. If $|a-b|<\frac{1}{4}$ then the probability that $a<\frac{1}{2}<b$ is

(a) $\frac{1}{11}$ (b) $\frac{1}{12}$ (c) $\frac{1}{13}$ (d) $\frac{1}{14}$

15. Mr X has a 75% chance of attending the annual meet. Miss Y has an 80% chance, if Mr X also attends. Otherwise she has a 50% chance of attending. If I go to the meet and see Miss Y there, then the probability that Mr X is also there is

(a) $\frac{24}{29}$ (b) $\frac{25}{29}$ (c) $\frac{26}{29}$ (d) $\frac{27}{29}$

16. A deck of 40 cards consists of four 1's, four 2's, ... and four 10's. A matching pair (2 cards with same number) is removed from the deck. These cards are not replaced back. The probability that 2 randomly selected cards also form a pair is $\frac{k}{^{38}C_2}$ where k =

(a) 54 (b) 55 (c) 64 (d) 65

17. Club X is in soccer league with 6 other teams, each of which it plays once. In any of its 6 matches, the chance that club X will win, lose or tie are each 1/3. The chance that club X will finish the season with more wins than losses is $\frac{k}{243}$ where k =

(a) 96 (b) 98 (c) 102 (d) 105

18. When rolling a certain biased 6-sided die with faces numbered 1, 2, 3, 4, 5, 6, the probability of obtaining face F is greater than 1/6, the chance of obtaining the face opposite < 1/6 and the chance of obtaining any one of the other faces is 1/6 and the sum of the numbers on the opposite faces is 7. When two such dice are rolled, the chance of getting a sum of 7 is $\frac{47}{288}$ then the chance of obtaining face F is

(a) $\frac{3}{24}$ (b) $\frac{4}{24}$ (c) $\frac{5}{24}$ (d) $\frac{6}{24}$

19. I arrive at an airport which has 12 gates arranged in a straight line with exactly 100 ft between adjacent gates. The departure gate is assigned at random. After waiting at that gate, I am told that it has been changed to a different gate, again at random. The probability that I have to walk 400 ft or less is

(a) $\frac{17}{33}$ (b) $\frac{19}{33}$ (c) $\frac{18}{35}$ (d) $\frac{17}{35}$

20. A box contains 4 balls, 1 red, 1 green and 2 black. Consider a game in which the player begins by drawing a ball from the box at random. If red ball is drawn, the game is over and the player wins ₹ 100. If black ball is drawn, the game is over and player wins nothing. If green ball is drawn, the ball is returned to the bag and the game continues. The chance that the player will win ₹ 100 in this game is

(a) 1/2 (b) 1/3 (c) 1/4 (d) 1/6

SOLUTIONS

1. (a) : $30^{39} = 2^{39} \times 3^{39} \times 5^{39}$

So, 40 × 40 × 40 divisors in total.

$2^a\, 3^b\, 5^c$ be a multiple of 30^{29} we must have $a \in [29, 39]$, $b \in [29, 39]$, $c \in [29, 39]$ *i.e.*, 11 choices for each a, b, c. Hence required probability $= \frac{11\times11\times11}{40\times40\times40} = \left(\frac{11}{40}\right)^3$

2. (c): There are $^{40}C_4$ ways of drawing 4 slips and $^{10}C_2$ ways to choose a and b. After a and b have been chosen, there are $^4C_2$ ways of choosing 2 slips labelled a and $^4C_2$ ways of choosing 2 slips labelled b.

So required probability $= \frac{^{10}C_2 \times {}^4C_2 \times {}^4C_2}{^{40}C_4} = \frac{36 \times {}^{10}C_2}{^{40}C_4}$

3. (d) : There are $^{10}C_2 = 45$ line segments, so there are $^{45}C_4$ ways of choosing four segments. Now, we count the number of ways of choosing four segments such that three of them form a triangle. There are $^{10}C_3$ ways of choosing 3 vertices. Since, we have 45 line segments from which 3 segments have already chosen. So there are remaining 42 ways of choosing the 4th segment. So, favourable ways $= {}^{10}C_3 \times 42 = 5040$

Hence, required probability $= \frac{5040}{^{45}C_4}$

4. (a) : Let the 2 points selected at random be $P(a, b)$ and $Q(c, d)$. So, area of $\Delta OPQ = \frac{1}{2}|ad-bc|$.

So, for area to be integer, $|ad - bc|$ must be even and so both (ad) and (bc) must be either even or both odd.

Case-I : When both ad and bc are odd, then probability that they are all odd $= \frac{4}{8}\times\frac{4}{8}\times\frac{4}{8}\times\frac{4}{8}=\frac{1}{16}$

Case-2 : When both are even then,

$$a_ed_o + a_od_e + a_ed_e = \frac{4}{8}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{4}{8}+\frac{4}{8}\cdot\frac{4}{8}=\frac{3}{4}$$

Similarly for $bc = \frac{3}{4}$

So, probability that both ad and bc are even $= \frac{9}{16}$

Case-1 and Case-2 are mutually exclusive.

So, required probability $= \frac{1}{16}+\frac{9}{16}=\frac{10}{16}=\frac{5}{8}$

5. (c): Let $[\log_{10}4x] = k = [\log_{10}x]$ then $x \in \left[10^k, \frac{10^{k+1}}{4}\right]$

Now, for $k > 0$ and $k = 0$, the interval $\left(10^k, \frac{10^{k+1}}{4}\right)$ is beyond the interval of (0, 1). But for $k < 0$, the whole interval is in (0, 1). So favourable x is obtained for $k = -1, -2, -3, ...$

For $k=-1,\ x\in\left[\frac{1}{10},\frac{1}{4}\right)$, for $k=-2,\ x\in\left[\frac{1}{100},\frac{1}{40}\right)$....

Sum of lengths of these intervals

$$=\left(\frac{1}{4}-\frac{1}{10}\right)+\left(\frac{1}{40}-\frac{1}{100}\right)+\ldots=\frac{3}{20}\left(1+\frac{1}{10}+\frac{1}{100}+\ldots\right)=\frac{1}{6}$$

Hence, required probability $= \frac{1/6}{1-0}=\frac{1}{6}$

6. (a) : Let the 4 persons were A, B, C, D.

Number of ways choosing the two persons which were shot $= {}^4C_2$

Let C, D were shot.

Then $C \leftrightarrow D$ and $A \to C$ or D and $B \to C$ or D.

Any person shooting any other person has probability = 1/3

So, probability that C, D were shot

$$=\underbrace{\left(\frac{1}{3}\right)\times\left(\frac{1}{3}\right)}_{\text{(for } C, D)}\times\underbrace{\frac{2}{3}\times\frac{2}{3}}_{\text{(for } A,B)}=\frac{4}{81}$$

Hence, probability that exactly 2 persons were shot

$$= {}^4C_2\times\frac{4}{81}=\frac{8}{27}$$

7. (b) : $P\left(\frac{\bar{E}\cap F}{\bar{G}}\right)=\frac{P(\bar{E}\cap F\cap\bar{G})}{P(\bar{G})}$

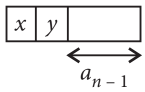
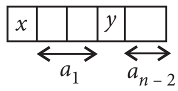
$$=\frac{1}{P(\bar{G})}[P(F)-P(F\cap E)-P(F\cap G)+P(E\cap F\cap G)]$$

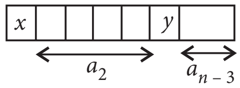
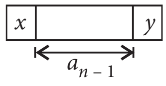
$$=\frac{1}{P(\bar{G})}[P(F)(1-P(E)-P(G))] = P(F)-\frac{P(E\cap F)}{P(\bar{G})}$$

8. (d) : Total number of arrangements of 8 children (holding 2 types of coins) $= \frac{8!}{4!\,4!}=70$

Let a_n denote the number of ways of arranging $2n$ children out of which n have one ₹ 1 coin and rest n have ₹ 2 coin such that no one waits for change at the counter.

Let x denote the child having ₹ 1 coin and y denote child having ₹ 2 coin in hand.

Case-1  Case-2 

Case-3  ... Case-n 

So, $a_n = a_{n-1} + a_1a_{n-2} + a_2a_{n-3} + ... + a_{n-1}$

and in particular $a_1 = 1$ $\boxed{x}\boxed{y}$

$a_2 = 2$ $\boxed{x}\boxed{x}\boxed{y}\boxed{y}$ or $\boxed{x}\boxed{y}\boxed{x}\boxed{y}$

So, $a_3 = 2 + 1 + 2 = 5$ and $a_4 = 14$

Hence, required probability =14/70 =1/5

9. (c): Total number of ways = no. of non-negative integers solutions to the equation

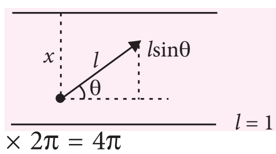
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$ which is ${}^{20}C_5$

Next of nobody is sitting next to anybody else, the number of empty chairs in the 4 middle slots must be positive *i.e.* $x_2, x_3, x_4, x_5 > 0$. Putting $y_1 = x_1 + 1$, $y_6 = x_6 + 1$, we have possible ways to $y_1 + x_2 + x_3 + x_4 + x_5 + y_6 = 17$ as ${}^{16}C_5$.

Hence required probability $= {}^{16}C_5/{}^{20}C_5$.

10. (b) : There are ${}^{n-6}C_7$ ways to draw 7 balls so that no two balls are consecutive and $(n-6)^{n-7}C_5$ ways to draw 7 balls so that there is exactly one pair of consecutive balls. Hence, ${}^{n-6}C_7 = (n-6)\,{}^{n-7}C_5$ gives $n = 54$

11. (a) : Consider the directed needle of length l.



Its position is determined by variables (x, θ). The pair (x, θ) is randomly chosen from the region $[0, 2) \times [0, 2\pi)$ with area $= 2 \times 2\pi = 4\pi$

The probability that needle will cross the upper horizontal line $= \frac{1}{4\pi}\int_0^{\pi}\int_0^{l\sin\theta} dx\, d\theta=\frac{1}{2\pi}$.

This is also the probability that it crosses the lower horizontal line.

So, the net probability that it crosses either the upper or lower horizontal lines = $1/\pi$.

12. (b) : Let as define the probability polynomial

$$P(x)=\left(\frac{1}{n}\cdot x+\frac{n-1}{n}\right)\left(\frac{2}{n}\cdot x+\frac{n-2}{n}\right)\left(\frac{3}{n}x+\frac{n-3}{n}\right)$$

Then x coefficient are probability of calculus questions. We require the constant term in the above polynomial $P(x)$.

Hence required probability $= \frac{(n-1)}{n} \cdot \frac{(n-2)}{n} \cdot \frac{(n-3)}{n}$.

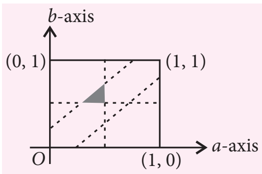
13. (c): There are 5 × 5! permutations with the first term not 1 and there are 4 × 4! permutations with third term 3 and first term not 1.

Hence required probability $= \frac{4 \times 4!}{5 \times 5!} = \frac{4}{25}$

14. (d) : Required probability

= Shaded area

$= \frac{(1/2)\times(1/4)\times(1/4)}{1-\frac{9}{16}} = \frac{1}{14}$



15. (a) : $P(X/Y) = \frac{P(Y/X)\cdot P(X)}{P(\text{Total})}$ (By Baye's theorem)

$= \frac{(3/4)\times(4/5)}{(3/4)\times(4/5)+(1/4)\times(1/2)} = \frac{24}{29}$

16. (b) : There are ${}^{38}C_2$ ways we can draw 2 cards from the deck. The two cards will form a pair if both are one of the 9 numbers that were not removed which can happen in 9 × ${}^{4}C_2$ ways, or if the two cards are the remaining 2 cards of the number that was removed, which can happen in 1 way.

So, required probability $= \frac{1+9\times {}^{4}C_2}{{}^{38}C_2} = \frac{55}{{}^{38}C_2}$

17. (b) : Probability (Number of wins > Number of losses) = Probability (Number of wins < Number of losses)

The only other possibility is, having same number of wins and losses.

So, by complement principle, the required probability is half the probability that club X does not have the same number of wins and losses. Now, we count the cases where same number of wins and losses are possible.

Case-1	0 ties, 3 wins, 3 losses	Probability $= \frac{6!}{3!3!} \times \frac{1}{3^6}$
Case-2	2 ties, 2 wins, 2 losses	Probability $= \frac{6!}{2!2!2!} \times \frac{1}{3^6}$
Case-3	4 ties, 1 win, 1 loss	Probability $= \frac{6!}{4!1!1!} \times \frac{1}{3^6}$
Case-4	6 ties	Probability $= \frac{6!}{6!} \times \frac{1}{3^6}$
		Total $= \frac{141}{3^6}$

So, required probability $= \frac{1-\frac{141}{3^6}}{2} = \frac{98}{243}$

18. (c) : Let as assume that face F has a 6 and opposite face has 1.

Let $A(n)$ be the probability of rolling a number n on one die and $B(n)$ be the probability of rolling a number n on other die. 7 can be obtained by rolling (2, 5), (5, 2), (3, 4), (4, 3).

Each has probability $= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

So, total $= 4 \times \frac{1}{36} = \frac{1}{9}$.

$\Rightarrow \frac{47}{288} - \frac{1}{9} = \frac{15}{288} = \frac{5}{96}$ is the chance of getting a 1 on die A and 6 on die B or 6 on die A and 1 on die B.

i.e., $A(1)\cdot B(6) + A(6)\cdot B(1) = \frac{5}{96}$

Since the two dice are identical, $B(1) = A(1)$, $B(6) = A(6)$

Also, $A(2) = A(3) = A(4) = A(5) = \frac{1}{6}$.

and $A(1) + A(2) + A(3) + A(4) + A(5) + A(6) = 1$

Solving, we have

$$A(6) = \frac{5}{24}, \frac{1}{8} \text{ but since, } A(6) > \frac{1}{6}$$

So, $A(6) = \frac{5}{24}$

19. (b) : There are in total 12 × 11 = 132 possible gate assignments that can be given.

Next we need to count the favourable cases.

Let the gates be $g_1, g_2, \ldots, g_{12}$

g_1 and g_{12} have 4 gates within 400 ft.

g_2 and g_{11} have 5 gates within 400 ft.

g_3 and g_{10} have 6 gates within 400 ft.

g_4 and g_9 have 7 gates within 400 ft.

g_5 through g_8 have 8 gates.

g_6 through g_7 have 8 gates.

So favourable cases in total are

2(4 + 5 + 6 + 7 + 8 + 8) = 76

Hence, required probability $= \frac{76}{132} = \frac{19}{33}$

20. (b) : Probability of win on 1st draw $= P(R) = \frac{1}{4}$

Probability of win on 2nd draw $= P(G) \times P(R) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$

Probability of win on 3rd draw

$$= P(G) \times P(G) \times P(R) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{64}$$

So, total probability of wining in finite number of draws

$$= \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1/4}{1-1/4} = \frac{1}{3}$$

MPP-7 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Differential Equations | Probability

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. The solution of differential equation
$\sec^2 x \tan y dx + \sec^2 y \tan x\, dy = 0$
$\forall x \in R - (2n+1)\frac{\pi}{2},\ n \in I$ is
(a) $\tan x = \tan y$
(b) $\tan x + \tan y = c,\ \forall x \in R$
(c) $\tan x \tan y = c$
(d) None of these

2. A random variable X has the probability distribution

X	1	2	3	4	5	6	7	8
$P(X)$	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

for the event $E = \{X$ is a prime number$\}$ and $F = \{X < 4\}$, then the probability $P(E \cup F)$ is
(a) 0.35 (b) 0.77 (c) 0.87 (d) 0.50

3. The solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)} \text{ is}$$

(a) $x\phi\left(\frac{y}{x}\right) = k$ (b) $\phi\left(\frac{y}{x}\right) = kx$
(c) $y\phi\left(\frac{y}{x}\right) = k$ (d) $\phi\left(\frac{y}{x}\right) = ky$

4. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing is
(a) $\frac{10}{3^5}$ (b) $\frac{17}{3^5}$ (c) $\frac{13}{3^5}$ (d) $\frac{11}{3^5}$

5. The differential equation satisfied by the curve $e^{2y} + 2axe^y + a^2 = 0$, where a is a parameter is
(a) $(1 - x^2)y_1^2 = 1$ (b) $(x^2 - 1)y_1^2 = 1$
(c) $(1 + x^2)y_1^2 = 1$ (d) $(x^2 + x)\, y_1^2 = y$

6. In a binomial distribution $B\left(n,\ p = \frac{1}{4}\right)$, if the probability of at least one success is greater than or equal to $\frac{9}{10}$, then n is greater than
(a) $\frac{1}{\log_{10} 4 + \log_{10} 3}$ (b) $\frac{9}{\log_{10} 4 - \log_{10} 3}$
(c) $\frac{4}{\log_{10} 4 - \log_{10} 3}$ (d) $\frac{1}{\log_{10} 4 - \log_{10} 3}$

One or More Than One Option(s) Correct Type

7. The differential equation representing the family of curves $y^2 = 2c\left(x + \sqrt{c}\right)$, where c is a positive parameter, is of
(a) order 1 (b) order 2
(c) degree 3 (d) degree 4

8. Let $0 < P(A) < 1$, $0 < P(B) < 1$ and $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$ then
(a) $P(B/A) = P(B) - P(A)$
(b) $P(A' - B') = P(A') - P(B')$
(c) $P(A \cup B)' = P(A')\, P(B')$
(d) $P(A/B) = P(A)$.

9. If A and B are two events such that $P(A \cup B) \geq \frac{3}{4}$ and $\frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$ then
(a) $P(A) + P(B) \geq \frac{7}{8}$ (b) $P(A) + P(B) \leq \frac{11}{8}$
(c) $P(A)\, P(B) \leq \frac{3}{8}$ (d) None of these

10. The solution of $\dfrac{xdx + ydy}{xdy - ydx} = \sqrt{\dfrac{a^2 - x^2 - y^2}{x^2 + y^2}}$ is

(a) $\sqrt{x^2 + y^2} = a(\sin(\tan^{-1}(y/x)) + \text{const.})$

(b) $\sqrt{x^2 + y^2} = a(\cos(\tan^{-1}(y/x)) + \text{const.})$

(c) $\sqrt{x^2 + y^2} = a(\tan(\sin^{-1}(y/x)) + \text{const.})$

(d) $y = x\tan\left(\text{const.} + \sin^{-1}\dfrac{1}{a}\sqrt{x^2 + y^2}\right)$

11. For any two events A and B in a sample space

(a) $P\left(\dfrac{A}{B}\right) \geq \dfrac{P(A)+P(B)-1}{P(B)}, P(B) \neq 0$ is always true.

(b) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$ does not hold.

(c) $P(A \cup B) = 1 - P(\bar{A})\ P(\bar{B})$, if A and B are independent.

(d) $P(A \cup B) = 1 - P(\bar{A})\ P(\bar{B})$, if A and B are disjoint.

12. The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c respectively of these subjects, the student has a 75% chance of passing in at least one, a 50% chance of passing in atleast two and a 40% chance of passing in exactly two. Which of the following relations is/are true?

(a) $p + m + c = \dfrac{19}{20}$ (b) $p + m + c = \dfrac{27}{20}$

(c) $pmc = \dfrac{1}{10}$ (d) $pmc = \dfrac{1}{4}$

13. If $y = ae^{-1/x} + b$ is a solution of $\dfrac{dy}{dx} = \dfrac{y}{x^2}$, then possible values of a and b are

(a) $a = 2, b = 0$ (b) $a = 5, b = 0$

(c) $a = -2, b = 0$ (d) $a = 1, b = 1$

Comprehension Type

A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required.

14. The probability that $X \geq 3$ equals

(a) $\dfrac{125}{216}$ (b) $\dfrac{25}{36}$ (c) $\dfrac{5}{36}$ (d) $\dfrac{25}{216}$

15. The conditional probability that $X \geq 6$ given $X > 3$ equals

(a) $\dfrac{125}{216}$ (b) $\dfrac{25}{216}$ (c) $\dfrac{5}{36}$ (d) $\dfrac{25}{36}$

Matrix Match Type

16. Match the following:

	Column I		Column II
P.	Solution of $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$, is	1.	$x^2 - y = -1$
Q.	If $x^2\,(ydx - xdy) = y^2(xdy - ydx)$ and $y(1) = 1$, then the equation is	2.	$x^2 - y^2 = c$
R.	If $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$, then the equation is	3.	$x^3 \cdot y^2 + \dfrac{x^2}{y} = c$
S.	Solution of $\left(\dfrac{dy}{dx}\right)^2 - x\dfrac{dy}{dx} - 2y + 2 = 0$, is	4.	$x^2 - y^2 = 0$

	P	Q	R	S
(a)	2	1	3	4
(b)	3	4	2	1
(c)	3	2	4	1
(d)	1	2	3	4

Integer Answer Type

17. If $f(x)$ is differentiable, then the solution of $dy + (yf'(x) - f'(x)f(x))dx = 0$ is $y = f(x) - k + ce^{-f(x)}$, then k is

18. The probability that in the toss of two dice we obtain an even sum or a sum less than 5 is $\dfrac{5}{m}$. The value of m is

19. Let $y = (A + Bx)e^{3x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + m\dfrac{dy}{dx} + ny = 0$, where $m, n \in I$, then $n =$

20. Two persons A and B have equal number of sons. There are three cinema tickets which are to be distributed among the sons of A and B. The probability that all the tickets go to sons of A is $\dfrac{1}{20}$. The number of sons to each of them is

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……

No. of questions correct ……

Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.
During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 169

JEE MAIN

1. The equation $z^3 + pz^2 + qz + r = 0$ has roots which are the vertices of an equilateral triangle then

(a) $p = 3q^2$ (b) $q = 3p^2$
(c) $p^2 = 3q$ (d) $q^2 = 3p$

2. Let $a, b \in N$. The number of pairs (a, b), $a < b$ such that $\frac{1}{a}+\frac{1}{b}=\frac{1}{2013}$ is

(a) 11 (b) 13 (c) 17 (d) 21

3. If $x-\frac{1}{x}=i$, then $x^{2013}+x^{-2013}=$

(a) 0 (b) 2 (c) $2i$ (d) $-2i$

4. Let $\{x\} = x - [x]$. The least integer greater than $\int_0^{100}\{\sqrt{x}\}dx$ is

(a) 50 (b) 51 (c) 52 (d) 53

5. A ray of light moving parallel to the y-axis gets reflected from a parabolic mirror whose equation is $(x - 2)^2 = 4(y + 1)$, passes through the point

(a) (0, 2) (b) (2, 0) (c) (–1, 2) (d) (2, –1)

JEE ADVANCED

6. If the equation $ax^2 + y^2 + bz^2 + 2yz + zx + 3xy = 0$ represents a pair of perpendicular planes, then $a =$

(a) $-\frac{7}{4}$ (b) $-\frac{3}{4}$ (c) 1 (d) 2

COMPREHENSION

$ABCDEF$ is a hexagon with all angles equal. $AB = 1$, $BC = 4$, $CD = 2$, $DE = 4$.

7. The perimeter of the hexagon is

(a) 15 (b) 16 (c) 17 (d) 18

8. The area of the hexagon is

(a) $\frac{23}{2}\sqrt{3}$ (b) $13\sqrt{3}$
(c) $\frac{43}{2}\sqrt{3}$ (d) $\frac{43}{4}\sqrt{3}$

INTEGER MATCH

9. A candidate appears for an examination consisting of 4 papers A, B, C and D. The maximum marks for each of first 3 papers is 10 and that of paper D is 20. If N is the number of ways of getting a total of 30 marks, then the sum of the digits of N is

MATRIX MATCH

10. Match the following columns:

	Column I		Column II
P.	If $f(x)=x^{\alpha}\sin\left(\frac{1}{x}\right), x \neq 0$, $f(0) = 0$, then $f'(x)$ is continuous for $\alpha =$	1.	0
Q.	If the curve $y = ax^2 + bx + c$ passes through the point (1, 2) and is tangent to the line $x = y$ at the origin, then $y(-1) =$	2.	1
R.	If $f(x)=\sin x+\int_0^{\pi/2}\sin x\cos t\,dt$, then $\int_0^{\pi/2}f(x)dx=$	3.	2
S.	The area bounded by the common tangents to the curves $x^2 = 2(y^2 + 1)$ and $y^2 = 2(x^2 + 1)$ (in sq. units) is	4.	3

	P	Q	R	S
(a)	4	3	2	1
(b)	4	1	3	3
(c)	3	1	2	4
(d)	1	2	3	4

See Solution Set of Maths Musing 168 on page no. 89

WB JEE MOCK TEST PAPER

Series-6

The entire syllabus of Mathematics of WB-JEE is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given bellow.

Unit No.	Topic	Syllabus In Details
UNIT NO. 6	Differential Calculus	Derivative of the sum, difference, product and quotient of two functions, chain rule, derivatives of polynomial, rational, trigonometric, inverse trigonometric, exponential and logarithmic functions. Derivative of implicit functions. Derivative up to order two,
	Trigonometry	Inverse trigonometrical functions and their properties. Height and distance.
	Vector Algebra	Vectors and scalars, addition of vectors, components of a vector in two dimensions and three dimensional space, scalar and vector products, scalar and vector triple product.
	Matrices and Determinants	Matrices as a rectangular array of real numbers, equality of matrices, addition, multiplication by a scalar and product of matrices, transpose of a matrix, determinant of square matrix of order up to three, properties of determinants, area of triangles using determinants , inverse of a square matrix of order up to three, properties of these matrix operations, diagonal, symmetric and skew-symmetric matrices and their properties, solutions of simultaneous linear equations in two or three variables.

Time : 1 hr 15 min. **Full marks : 50**

CATEGORY-I

For each correct answer one mark will be awarded, whereas, for each wrong answer, 25% of total marks (1/4) will be deducted. If candidates mark more than one answer, negative marking will be done.

1. If $A=\begin{bmatrix}0 & 3\\ 4 & 5\end{bmatrix}$ and $kA=\begin{bmatrix}0 & 4a\\ 3b & 60\end{bmatrix}$, then the value of k, a, b are respectively
   (a) 12, 9, 16 (b) 9, 12, 16
   (c) 12, 9, 15 (d) 16, 9, 12

2. If $A=\begin{bmatrix}1 & 2\\ 2 & 3\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$ then
   (a) both AB and BA exist.
   (b) AB exist but BA does not exist.
   (c) neither AB nor BA exist.
   (d) AB does not exist but BA exists.

3. If $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$, then $A^2-4A=$
   (a) I (b) $-I$ (c) $3I$ (d) $-3I$

4. If for matrix A, $A^3=0$ then $A^2+A+I=$
   (a) $I+A$ (b) $I-A$
   (c) $(I+A)^{-1}$ (d) $(I-A)^{-1}$

5. If $A^2-A+I=0$ then the inverse of matrix A is
   (a) $A+I$ (b) A (c) $A-I$ (d) $I-A$

6. If the inverse matrix A^{-1} of matrix A exists then $\det(A^{-1})$ will be
   (a) $\det(A)$ (b) $-\det(A)$
   (c) $\dfrac{1}{\det(A)}$ (d) None of these

7. If $A=\begin{bmatrix}1 & 2\\ -4 & -1\end{bmatrix}$ then $A^{-1}=$
   (a) $\frac{1}{7}\begin{bmatrix}-1 & 2\\ 4 & 1\end{bmatrix}$ (b) $\frac{1}{7}\begin{bmatrix}1 & 2\\ -4 & -1\end{bmatrix}$
   (c) $\frac{1}{7}\begin{bmatrix}-1 & -2\\ 4 & 1\end{bmatrix}$ (d) does not exist

By : Sankar Ghosh, S.G.M.C, Kolkata, Ph: 09831244397.

8. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then (x, y) =

(a) (3, 1) (b) (1, 3) (c) (0, 3) (d) (0, 0)

9. If $\begin{vmatrix} \alpha^2+2\alpha & 2\alpha+1 & 1 \\ 2\alpha+1 & \alpha+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (\alpha-1)^p$ then the value of p is

(a) 1 (b) 2 (c) 3 (d) 4

10. If the three points $(3q, 0)$, $(0, 3p)$ and $(1, 1)$ are collinear then which one of the following is true?

(a) $\frac{1}{p}+\frac{1}{q}=1$ (b) $\frac{3}{p}+\frac{1}{q}=1$

(c) $\frac{1}{p}+\frac{1}{q}=3$ (d) $\frac{1}{p}+\frac{3}{q}=1$

11. If $y = \sin\left(\frac{\pi}{6}e^{xy}\right)$, then the value of $\frac{dy}{dx}$ at $x = 0$, is

(a) $\frac{\pi\sqrt{3}}{6}$ (b) $\frac{\pi\sqrt{3}}{12}$ (c) $\frac{\pi\sqrt{3}}{18}$ (d) $\frac{\pi\sqrt{3}}{24}$

12. The differential coefficient of $f(\log_e x)$ where $f(x) = \log_e x$ is

(a) $\frac{x}{\log_e x}$ (b) $(x\log_e x)^{-1}$

(c) $\frac{\log_e x}{x}$ (d) $x\log_e x$

13. If $y = 2\cos2\theta(1 - \cos2\theta)$ and $x = 3\sin2\theta(1 + \cos2\theta)$ then the value of $\frac{dy}{dx}$ is

(a) $\frac{3}{2}\cot\theta$ (b) $\frac{2}{3}\cot\theta$

(c) $\frac{2}{3}\tan\theta$ (d) $\frac{3}{2}\tan\theta$

14. If $x = e^{y+e^{y+\ldots\infty}}, x > 0$, then $\frac{dy}{dx}$ =

(a) $\frac{1-x}{x}$ (b) $\frac{x}{1+x}$ (c) $\frac{1+x}{x}$ (d) $\frac{1}{x}$

15. If $x^m \cdot y^n = (x+y)^{m+n}$, then $\frac{dy}{dx}$ =

(a) $\frac{y}{x}$ (b) $\frac{x+y}{xy}$ (c) xy (d) $\frac{x}{y}$

16. $y = (1+x)(1+x^2)(1+x^4)\ldots(1+x^n)$ then $\left(\frac{dy}{dx}\right)_{x=0}$ =

(a) 0 (b) –1 (c) 1 (d) 2

17. Differential coefficient of e^{x^2} with respect to $\log x^2$ is

(a) e^{x^2} (b) xe^{x^2}

(c) $x^2e^{x^2}$ (d) $2x^2e^{x^2}$

18. The second order derivative of $a\sin^3 t$ with respect to $a\cos^3 t$ at $t = \pi/4$ is

(a) 2 (b) $\frac{1}{12a}$ (c) $\frac{4\sqrt{2}}{3a}$ (d) $\frac{3a}{4\sqrt{2}}$

19. If $y = e^{m\sin^{-1}x}$, then $\frac{d^2y}{dx^2}$ at $x = 0$ is

(a) m (b) m^2 (c) $-m^2$ (d) $2m$

20. If $y = \sin(2\sin^{-1}x)$, then it satisfies the differential equation

(a) $(1-x^2)y_2 - xy_1 + 4y = 0$

(b) $(1+x^2)y_2 - xy_1 + 4y = 0$

(c) $(1-x^2)y_2 - xy_1 + y = 0$

(d) $(1+x^2)y_2 - xy_1 + y = 0$

21. The principal value of $\sin^{-1}\tan\left(-\frac{5\pi}{4}\right)$ is

(a) $\pi/4$ (b) $-\pi/4$ (c) $\pi/2$ (d) $-\pi/2$

22. The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$ has a solution for

(a) $|a| \ge \frac{1}{\sqrt{2}}$ (b) $\frac{1}{2} \le |a| < \frac{1}{\sqrt{2}}$

(c) all real values of a

(d) $|a| \le \frac{1}{\sqrt{2}}$

23. $2\tan^{-1}(\text{cosec}(\tan^{-1}x) - \tan(\cot^{-1}x))$ =

(a) $\tan^{-1}x$ (b) $\cot^{-1}x$

(c) 0 (d) 1

24. The value of $\cos\left[2\tan^{-1}\frac{1+x}{1-x} + \sin^{-1}\frac{1-x^2}{1+x^2}\right]$ is

(a) $\sqrt{2}$ (b) 1 (c) 0 (d) –1

25. $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = \frac{\pi^3}{32}$ if x equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) 1

26. The vectors $\overrightarrow{AB} = 3\hat{i} + 4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ are two sides of a triangle ABC. Then length of the median through A is

(a) $\sqrt{18}$ (b) $\sqrt{72}$ (c) $\sqrt{33}$ (d) $\sqrt{288}$

27. If $\vec{\alpha}=2\hat{i}+\hat{j}+\hat{k}, \vec{\beta}=\hat{i}-2\hat{j}+2\hat{k}$ and $\vec{\gamma}=3\hat{i}-4\hat{j}+2\hat{k}$ then the projection of $(\vec{\alpha}+\vec{\gamma})$ in the direction of $\vec{\beta}$ is

(a) $\frac{17}{3}$ (b) $\frac{5}{3}$ (c) $\frac{17}{\sqrt{43}}$ (d) $\frac{14}{3}$

28. If $|\vec{a}|=4$, $|\vec{b}|=2$ and the angle between the vectors $\vec{a}$ and $\vec{b}$ is $\pi/6$, then $(\vec{a}\times\vec{b})^2=$

(a) 48 (b) 16 (c) 0 (d) 3

29. If $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}\neq 0$, then $\vec{a}+\vec{b}+\vec{c}=$

(a) $-\vec{b}$ (b) $\vec{0}$ (c) $2\vec{a}$ (d) $3\vec{b}$

30. The value of λ(constant), when $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+\lambda\hat{j}+5\hat{k}$ be coplanar, is

(a) 2 (b) -4 (c) 4 (d) -2

CATEGORY-II

Every correct answer will yield 2 marks. For incorrect response, 25% of full mark (1/2) would be deducted. If candidates mark more than one answer, negative marking will be done.

31. If $f(x)=\begin{vmatrix} x & \sin x & \cos x \\ x^2 & \tan x & -x^3 \\ 2x & \sin 2x & 5x \end{vmatrix}$ then $\lim_{x\to 0}\frac{f'(x)}{x}$ is

(a) 1 (b) 4 (c) 3 (d) -4

32. The derivative of the function

$$f(x)=\cos^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x-3\sin x)\right\}+\sin^{-1}\left\{\frac{1}{\sqrt{13}}(2\sin x+3\cos x)\right\}$$

with respect to $\sqrt{1+x^2}$ is

(a) $2x$ (b) $2\sqrt{1+x^2}$

(c) $\frac{2}{x}\sqrt{1+x^2}$ (d) $\frac{2x}{\sqrt{1+x^2}}$

33. If the unit vectors $\vec{a}$ and $\vec{b}$ are inclined at an angle 2θ and $|\vec{a}-\vec{b}|$ is less than 1, if $0\le\theta\le\pi$, θ lies in the interval

(a) $\left[0,\frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6},\pi\right]$ (b) $\left(0,\frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6},\pi\right)$

(c) $\left[\frac{\pi}{6},\frac{\pi}{2}\right]$ (d) $\left[\frac{\pi}{2},\frac{5\pi}{6}\right]$

34. Let $A=\begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$. If $|A^2|=25$, then $|\alpha|=$

(a) 1/5 (b) 5

(c) 5^2 (d) 1

35. A piece of paper in the shape of a sector of a circle of radius 10 cm and of angle 216° just covers the lateral surface of a right circular cone of vertical angle 2θ. Then $\sin\theta$ is

(a) 3/5 (b) 4/5

(c) 3/4 (d) None of these

CATEGORY-III

In this section more than 1 answer can be correct. Candidates will have to mark all the correct answers, for which 2 marks will be awarded. If, candidates marks one correct and one incorrect answer then no marks will be awarded. But if, candidate makes only correct, without making any incorrect, formula below will be used to allot marks. 2 × (no. of correct response/total no. of correct options)

36. If $f(x)=\cos^{-1}x+\cos^{-1}\left\{\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right\}$ then

(a) $f\left(\frac{2}{3}\right)=\frac{\pi}{3}$ (b) $f\left(\frac{2}{3}\right)=2\cos^{-1}\frac{2}{3}-\frac{\pi}{3}$

(c) $f\left(\frac{1}{3}\right)=\frac{\pi}{3}$ (d) $f\left(\frac{1}{3}\right)=2\cos^{-1}\frac{1}{3}-\frac{\pi}{3}$

37. If $f(x)=2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$

(a) $f'(2)=f'(3)$ (b) $f'(2)=0$

(c) $f'\left(\frac{1}{2}\right)=\frac{16}{5}$ (d) $f'\left(\frac{1}{2}\right)=0$

38. The resolved part of the vector $\vec{a}$ along the vector $\vec{b}$ is $\vec{\lambda}$ and that perpendicular to $\vec{b}$ is $\vec{\mu}$. Then

(a) $\vec{\lambda}=\frac{(\vec{a}\cdot\vec{b})\vec{a}}{\vec{a}^2}$ (b) $\vec{\lambda}=\frac{(\vec{a}\cdot\vec{b})\vec{b}}{\vec{b}^2}$

(c) $\vec{\mu}=\frac{(\vec{b}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}}{\vec{b}^2}$

(d) $\vec{\mu}=\frac{\vec{b}\times(\vec{a}\times\vec{b})}{\vec{b}^2}$

39. Let $f(n)=\begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2} \\ {}^nC_n & {}^{n+1}C_{n+1} & {}^{n+2}C_{n+2} \end{vmatrix}$ then $f(n)$ is divisible by

(a) n^2+n+1 (b) $(n+1)!$

(c) $n!$ (d) None of these

40. $B=\begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$ is a

(a) rectangular matrix (b) singular matrix

(c) square matrix (d) nonsingular matrix.

SOLUTIONS

1. (a) : $A=\begin{bmatrix}0&3\\4&5\end{bmatrix}\Rightarrow kA=\begin{bmatrix}0&3k\\4k&5k\end{bmatrix}=\begin{bmatrix}0&4a\\3b&60\end{bmatrix}$

$\therefore$ $5k = 60 \Rightarrow k = 12$

$3k = 4a \Rightarrow 4a = 36 \Rightarrow a = 9$

Also, $3b = 4k = 48 \Rightarrow b = 16$

2. (b) : Here, $A=\begin{bmatrix}1&2\\2&3\\3&4\end{bmatrix}$ and $B=\begin{bmatrix}1&2\\2&1\end{bmatrix}$

The order of the matrix A is 3×2 and that of B is 2×2, therefore AB is defined but BA is not defined.

3. (d) : Here, $A=\begin{bmatrix}2&-1\\-1&2\end{bmatrix}$

$\therefore\ A^2-4A=\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}-4\begin{bmatrix}2&-1\\-1&2\end{bmatrix}$

$=\begin{bmatrix}5&-4\\-4&5\end{bmatrix}-\begin{bmatrix}8&-4\\-4&8\end{bmatrix}=\begin{bmatrix}-3&0\\0&-3\end{bmatrix}=-3\begin{bmatrix}1&0\\0&1\end{bmatrix}=-3I$

4. (a) : $A^2 + A + I = A^{-1}[A^3 + A^2 + A]$

$= A^{-1}[A^2 + A] = A + I$

5. (d) : $A^2 - A + I = 0 \Rightarrow A^{-1}(A^2 - A + I) = 0$

$\Rightarrow A^{-1}A^2 - A^{-1}A + A^{-1}I = 0$

$\Rightarrow A - I + A^{-1} = 0 \Rightarrow A^{-1} = I - A$

6. (c): $\because$ $|AB| = |A| \times |B|$ and $AA^{-1} = I$,

$\therefore\ |AA^{-1}|=|I|\Rightarrow |A|\times|A^{-1}|=1\Rightarrow |A^{-1}|=\frac{1}{|A|}.$

7. (c): If $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ then $A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

$\therefore$ If $A=\begin{bmatrix}1&2\\-4&-1\end{bmatrix}$ then $A^{-1}=\frac{1}{7}\begin{bmatrix}-1&-2\\4&1\end{bmatrix}$

8. (d) : Given that $\begin{vmatrix}6i&-3i&1\\4&3i&-1\\20&3&i\end{vmatrix}=x+iy$

Applying $R_1 \to R_1 + R_2$, we get $\begin{vmatrix}4+6i&0&0\\4&3i&-1\\20&3&i\end{vmatrix}=x+iy$

$\Rightarrow (4 + 6i)(0) = x + iy \Rightarrow x = 0, y = 0$

9. (c)

10. (c): Given that the points $(3q, 0)$, $(0, 3p)$ and $(1, 1)$ are collinear.

$\therefore\ \begin{vmatrix}3q&0&1\\0&3p&1\\1&1&1\end{vmatrix}=0\Rightarrow 3q(3p-1)+1(0-3p)=0$

$\Rightarrow 9pq - 3q - 3p = 0 \Rightarrow \frac{1}{p}+\frac{1}{q}=3$

11. (d) : Here, $y=\sin\left(\frac{\pi}{6}e^{xy}\right)$

$\therefore\ \frac{dy}{dx}=\cos\left(\frac{\pi}{6}e^{xy}\right)\times\frac{\pi}{6}e^{xy}\left(x\frac{dy}{dx}+y\right)$

Now putting $x = 0$ in the above equation we get

$\left(\frac{dy}{dx}\right)_{x=0}=\frac{\pi}{6}\times\frac{\sqrt{3}}{2}\left(0+y_{(x=0)}\right)$

$\Rightarrow \frac{dy}{dx}=\frac{\pi}{6}\times\frac{\sqrt{3}}{2}\times\frac{1}{2}=\frac{\pi\sqrt{3}}{24}\ \left(\because y_{(x=0)}=\sin\frac{\pi}{6}=\frac{1}{2}\right)$

12. (b) : $f(x) = \log_e x \Rightarrow f(\log_e x) = \log_e(\log_e x)$

$\therefore\ \frac{d\{f(\log_e x)\}}{dx}=\frac{1}{x\log_e x}$

13. (c): Given that

$y = 2\cos2\theta(1 - \cos2\theta)$ and $x = 3\sin2\theta(1 + \cos2\theta)$

Now $\frac{dy}{d\theta}=2\cos2\theta(2\sin2\theta)-2(1-\cos2\theta)(2\sin2\theta)$

$= 4(\sin4\theta - \sin2\theta) = 8\cos3\theta\sin\theta$

And $\frac{dx}{d\theta}=3\sin2\theta(-2\sin2\theta)+3(1+\cos2\theta)\times2\cos2\theta$

$= 6(\cos4\theta + \cos2\theta) = 12\cos3\theta\cos\theta$

$\therefore\ \frac{dy}{dx}=\frac{2}{3}\tan\theta$

14. (a) : We have, $x=e^{y+e^{y+\ldots\infty}}$

$\Rightarrow \log_e x = y + e^{y+\ldots\infty} = y + x$

$\therefore\ \frac{1}{x}=\frac{dy}{dx}+1\Rightarrow \frac{dy}{dx}=\frac{1-x}{x}$

15. (a) : $x^m \cdot y^n = (x + y)^{m+n}$

Taking logarithm on both sides

$m\log x + n\log y = (m + n)\log(x + y)$

Differentiating both sides w.r.t. x, we get

$\Rightarrow \frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=\frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$

$\Rightarrow \frac{nx-my}{y(x+y)}\times\frac{dy}{dx}=\frac{nx-my}{x(x+y)}\Rightarrow\frac{dy}{dx}=\frac{y}{x}$

16. (c): Here, we have

$y = (1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^n)$

Taking logarithm on both sides we get

$\log y = \log(1 + x) + \log(1 + x^2) + \ldots + \log(1 + x^n)$

Differentiating both sides w.r.t. x, we get

$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\ldots+\frac{nx^{n-1}}{1+x^n}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=0}=y_{(x=0)}\cdot1=1\cdot1=1$

17. (c): Let $u = e^{x^2}$ and $v = \log x^2$

Now $\frac{du}{dx} = 2xe^{x^2}$ and $\frac{dv}{dx} = \frac{2}{x} \Rightarrow \frac{du}{dv} = x^2 e^{x^2}$

18. (c)

19. (b) : $y = e^{m\sin^{-1}x} \Rightarrow \frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}} \times e^{m\sin^{-1}x}$

$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = me^{m\sin^{-1}x} = my \Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2$

$\Rightarrow 2(1-x^2)\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 2m^2y\frac{dy}{dx}$

$\Rightarrow 2m\left(\frac{d^2y}{dx^2}\right)_{x=0} = 2m^2\cdot 1\cdot m \quad \left(\because \text{at } x=0,\ y=1 \text{ and } \frac{dy}{dx} = m\right)$

$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=0} = m^2$

20. (a) : $y = \sin(2\sin^{-1}x) \Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = 2\cos(2\sin^{-1}x)$

$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4\{1-\sin^2(2\sin^{-1}x)\} = 4-4y^2$

$\Rightarrow 2(1-x^2)\left(\frac{dy}{dx}\right)\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -8y\frac{dy}{dx}$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$

21. (d) : $\sin^{-1}\tan\left(-\frac{5\pi}{4}\right) = \sin^{-1}\left(-\tan\frac{5\pi}{4}\right)$

$= \sin^{-1}\left\{-\tan\left(\pi+\frac{\pi}{4}\right)\right\} = \sin^{-1}\left(-\tan\frac{\pi}{4}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$

22. (d) : We know that $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} \le 2\sin^{-1}a \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} \le \sin^{-1}a \le \frac{\pi}{4}$

$\Rightarrow \sin\left(-\frac{\pi}{4}\right) \le a \le \sin\left(\frac{\pi}{4}\right) \Rightarrow -\frac{1}{\sqrt{2}} \le a \le \frac{1}{\sqrt{2}} \Rightarrow |a| \le \frac{1}{\sqrt{2}}$

23. (a) : Let $\tan^{-1}x = \alpha \quad \therefore \cot^{-1}x = \frac{\pi}{2} - \alpha$

$\therefore 2\tan^{-1}(\operatorname{cosec}(\tan^{-1}x) - \tan(\cot^{-1}x))$

$= 2\tan^{-1}\left(\operatorname{cosec}\alpha - \tan\left(\frac{\pi}{2}-\alpha\right)\right) = 2\tan^{-1}(\operatorname{cosec}\alpha - \cot\alpha)$

$= 2\tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right) = 2\tan^{-1}\left(\tan\frac{\alpha}{2}\right) = \alpha = \tan^{-1}x$

24. (d) : Let $x = \tan\alpha$, then we get

$\cos\left[2\tan^{-1}\left(\frac{1+\tan\alpha}{1-\tan\alpha}\right) + \sin^{-1}\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right)\right]$

$= \cos\left[2\tan^{-1}\tan\left(\frac{\pi}{4}+\alpha\right) + \sin^{-1}(\cos 2\alpha)\right]$

$= \cos\left(\frac{\pi}{2} + 2\alpha + \frac{\pi}{2} - 2\alpha\right) = \cos\pi = -1$

25. (b) : $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = \frac{\pi^3}{32}$

Let $\sin^{-1}x = \alpha \quad \therefore \cos^{-1}x = \frac{\pi}{2} - \alpha$

$\therefore \alpha^3 + \left(\frac{\pi}{2}-\alpha\right)^3 = \frac{\pi^3}{32}$

$\Rightarrow \frac{3\pi}{2}\alpha^2 - 3\left(\frac{\pi}{2}\right)^2\alpha + \frac{3\pi^3}{32} = 0$

$\Rightarrow \alpha^2 - \frac{\pi}{2}\alpha + \frac{\pi^2}{16} = 0 \Rightarrow \left(\alpha - \frac{\pi}{4}\right)^2 = 0$

$\Rightarrow \alpha = \frac{\pi}{4} \Rightarrow \sin^{-1}x = \frac{\pi}{4} \Rightarrow x = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

26. (c): From the geometry, we have $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$

$\Rightarrow \overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}) = \frac{1}{2}(8\hat{i} - 2\hat{j} + 8\hat{k})$

$\therefore |\overrightarrow{AD}| = \sqrt{4^2+1^2+4^2} = \sqrt{33}$

27. (a) : Here $\vec{\alpha} = 2\hat{i} + \hat{j} + \hat{k},\ \vec{\beta} = \hat{i} - 2\hat{j} + 2\hat{k}$

and $\vec{\gamma} = 3\hat{i} - 4\hat{j} + 2\hat{k}$. Now, $\vec{\alpha} + \vec{\gamma} = 5\hat{i} - 3\hat{j} + 3\hat{k}$

$\therefore$ Projection of $(\vec{\alpha} + \vec{\gamma})$ on $\vec{\beta}$

$= \frac{(\vec{\alpha}+\vec{\gamma})\cdot\vec{\beta}}{|\vec{\beta}|} = \frac{5\cdot 1 + (-2)(-3) + 2\cdot 3}{\sqrt{1^2+(-2)^2+2^2}} = \frac{17}{3}$

28. (b) : Here $|\vec{a}| = 4,\ |\vec{b}| = 2$

$\therefore |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta \ \left[\because |\hat{n}| = 1 \text{ and } 0 \le \theta \le \pi \therefore \sin\theta > 0\right]$

$\Rightarrow |\vec{a}\times\vec{b}|^2 = 4^2 \times 2^2 \times \sin^2\frac{\pi}{6} = 16\times 4\times\frac{1}{4} = 16$

29. (b) : We have, $\vec{a}\times\vec{b} = \vec{b}\times\vec{c}$

$\Rightarrow \vec{a}\times\vec{b} + \vec{c}\times\vec{b} = \vec{0} \Rightarrow (\vec{a}+\vec{c})\times\vec{b} = \vec{0}$

$\Rightarrow (\vec{a}+\vec{b}+\vec{c})\times\vec{b} = \vec{0} \quad (\because \vec{b}\times\vec{b} = \vec{0})$

Similarly, $\vec{b}\times\vec{c} = \vec{c}\times\vec{a} \Rightarrow (\vec{a}+\vec{b}+\vec{c})\times\vec{c} = 0$

$\because \vec{b}, \vec{c}$ are non-collinear $\therefore \vec{a}+\vec{b}+\vec{c} = 0$

30. (b) : Given that

$2\hat{i} - \hat{j} + \hat{k},\ \hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + \lambda\hat{j} + 5\hat{k}$ are coplanar.

$\therefore$ Their scalar triple product will be zero.

$\Rightarrow \begin{vmatrix} 3 & \lambda & 5 \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix} = 0 \Rightarrow 7\lambda + 28 = 0 \Rightarrow \lambda = -4$

31. (d)

32. (c): Here, $f(x)=\cos^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x-3\sin x)\right\}$

$$+\sin^{-1}\left\{\frac{1}{\sqrt{13}}(2\sin x+3\cos x)\right\}$$

Let $\frac{2}{\sqrt{13}}=\cos\theta$ and $\frac{3}{\sqrt{13}}=\sin\theta$

$\therefore$ $f(x) = \cos^{-1}(\cos\theta\cos x - \sin\theta\sin x)$ $+ \sin^{-1}(\cos\theta \sin x + \sin\theta\cos x)$

$= \cos^{-1}\cos(\theta + x) + \sin^{-1} \sin(\theta + x) = 2x + 2\theta$

$\therefore$ The required derivative is

$$\frac{df(x)}{d(\sqrt{1+x^2})}=\frac{2}{\frac{2x}{2\sqrt{1+x^2}}}=2\cdot\frac{\sqrt{1+x^2}}{x}$$

33. (a) : Given that $\vec{a}$ and $\vec{b}$ are unit vectors and angle between $\vec{a}$ and $\vec{b}$ is 2θ and $|\vec{a}-\vec{b}|<1$.

$\therefore$ $\left|\vec{a}-\vec{b}\right|^2=(\vec{a}-\vec{b})^2=|\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}$

$= 2(1 - \cos2\theta) = 4\sin^2\theta$

$\therefore$ $\left|\vec{a}-\vec{b}\right|=2|\sin\theta|\Rightarrow|\sin\theta|<\frac{1}{2}\Rightarrow\theta\in\left[0,\frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6},\pi\right]$

34. (a) : Let $A=\begin{bmatrix}5 & 5\alpha & \alpha\\ 0 & \alpha & 5\alpha\\ 0 & 0 & 5\end{bmatrix}$

$\therefore$ $|A^2|=|A|^2=\begin{vmatrix}5 & 5\alpha & \alpha\\ 0 & \alpha & 5\alpha\\ 0 & 0 & 5\end{vmatrix}^2=[5(5\alpha)]^2=(25\alpha)^2$

$\because$ $|A^2|=25$ $\therefore (25\alpha)^2=25\Rightarrow\alpha^2=\frac{1}{25}\therefore|\alpha|=\frac{1}{5}.$

35. (a) : The area of the sector $=\left(\frac{\pi\cdot10^2}{360}\times216\right)$ cm^2

= lateral surface area of the cone

The circumference of the base circle of the cone

$=\left(\frac{2\pi\times10}{360}\times216\right)$ cm $\therefore$ $2\pi r=\frac{2\pi\times10}{360}\times216\Rightarrow r=6$ cm.

The slant height of the cone = $(6\text{cosec}\theta)$ cm

$\therefore$ Lateral surface area = πrl = $(\pi \cdot 6 \cdot 6\text{cosec}\theta)$ cm^2

$\therefore$ $\pi\cdot36\text{cosec}\theta=\frac{\pi\cdot10^2\cdot216}{360}\Rightarrow\sin\theta=\frac{3}{5}.$

36. (a, d) : $f(x)=\cos^{-1}x+\cos^{-1}\left\{\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right\}$

$\therefore$ $f(x)=\cos^{-1}x+\cos^{-1}\left\{\frac{1}{2}\cdot x+\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right\}$

$=\cos^{-1}x\pm\left(\cos^{-1}\frac{1}{2}-\cos^{-1}x\right)$

(as $\cos^{-1}\frac{1}{2}>$ is either greater or less than $\cos^{-1}x$)

$$=\begin{cases}\cos^{-1}\frac{1}{2} & \left(\text{if } \cos^{-1}\frac{1}{2}>\cos^{-1}x\text{, which holds for } x=\frac{2}{3}\right)\\ 2\cos^{-1}x-\cos^{-1}\frac{1}{2} & \left(\text{if } \cos^{-1}\frac{1}{2}<\cos^{-1}x\text{, which holds for } x=\frac{1}{3}\right)\end{cases}$$

37. (a, b, c) : Given that,

$$f(x)=2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}\quad(\text{when } x\ \le 1)$$

Let $x = \tan\theta$

$\therefore$ $f(x) = 2\theta + 2\theta = 4\theta = 4\tan^{-1}x$

$\Rightarrow$ $f'(x)=\frac{4}{1+x^2}$ $\therefore$ $f'\left(\frac{1}{2}\right)=\frac{16}{5}$

But, if $x > 1$, $\sin^{-1}\frac{2x}{1+x^2}=\pi-2\tan^{-1}x$

$\therefore$ $f(x) = 2\tan^{-1}x + \pi - 2\tan^{-1}x = \pi$

$\Rightarrow$ $f'(x) = 0$ $\Rightarrow$ $f'(2) = f'(3)$

38. (b, c, d) : The magnitude of the resolved part along the vector $\vec{b}=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$\therefore$ The resolved part along the vector $\vec{b}$

$$=\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|}=\frac{(\vec{a}\cdot\vec{b})\vec{b}}{\vec{b}^2}$$

$\therefore$ The resolved part perpendicular to $\vec{b}$

$$=\vec{a}-\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|}=\frac{(\vec{b}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}}{\vec{b}^2}=\frac{\vec{b}\times(\vec{a}\times\vec{b})}{\vec{b}^2}$$

39. (a,c) : $f(n)=\begin{vmatrix}n & n+1 & n+2\\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2}\\ {}^nC_n & {}^{n+1}C_{n+1} & {}^{n+2}C_{n+2}\end{vmatrix}$

$$=\begin{vmatrix}n & n+1 & n+2\\ n! & (n+1)! & (n+2)!\\ 1 & 1 & 1\end{vmatrix}=n!\begin{vmatrix}n & n+1 & n+2\\ 1 & n+1 & (n+2)(n+1)\\ 1 & 1 & 1\end{vmatrix}$$

Applying $C_1\to C_1-C_2$ and $C_2\to C_2-C_3$, we get

$$f(n)=n!\begin{vmatrix}-1 & -1 & n+2\\ -n & -(n+1)^2 & (n+2)(n+1)\\ 0 & 0 & 1\end{vmatrix}=n!\{(n+1)^2-n\}$$

$= n!(n^2 + n + 1)$

40. (c, d) : $B=\begin{bmatrix}1 & -2 & 3\\ 2 & -1 & 4\\ 3 & 4 & 1\end{bmatrix}$

$\therefore$ $|B| = 1(-1 - 16) + 2(2 - 12) + 3(8 + 3) = -4$

◈◈

VIT VARSITY, VELLORE GETS 4-STAR RATING FOR OVERALL PERFORMANCE

VIT University, Vellore has become the first Indian university to get international recognition in terms of getting a 4-star rating for overall performance in a recently concluded audit of Quaquarelli Symonds (QS), UK.

A certificate to this effect was handed over to the Chancellor of University, Dr G Viswanathan here by QS South Asia Director, Ashwani Fernandes in presence of Minister of State for Finance Santosh Kumar Gangwar.

Speaking on the occasion, Mr Gangwar said that in deference to the wishes of Prime Minister Narendra Modi, the VIT university has gone ahead in promoting the meaningful education with wisdom and skill development.

VIT University, India became the first Indian university to receive a 4 Star rating for overall performance in a recently concluded audit of Quacquarelli Symonds (QS), UK.

They also became the first Indian university to receive 5 stars category rating in 5 different categories. The categories are Teaching, Innovation, Facilities, Employability and Inclusiveness. VIT is the only university in India now with 4 STAR overall rating of QS.

Quacquarelli Symonds (QS) from the UK, founded in 1990, is the world's leading network for top careers and education.

Selection Criteria

The core criteria for QS Star evaluation are – Teaching, Employability, Research and Internationalization. The other criteria are – Innovation, culture, access, engagement, facilities, online learning, Discipline ranking and Accreditation. A university needs to score 550 points out of 1000 to get 4 star rating. The prerequisites of 4 star rating is at least 75 academic referees or must have at least 2 citations per faculty member, another prerequisite is the university should have at least 1% international students. However, these prerequisite are not applicable for 3 star rating. These have made QS 4 star rating very challenging and highly competitive.

What this rating means for VIT University?

This 4 star rating is valid for 3 years. In the meantime VIT has already started its work towards 5 star rating for subsequent evaluation by increasing number of foreign studentsand also the number of faculty citations.

VIT has increased facilities in research and approaching the Government of India for more research funds. VIT University is also working to be one among the Top 200 universities of the world by next 5 years.

This achievement of VIT University is a significant milestone towards our Honourable Prime Minister's vision to upgrade 20 Indian Universities among the Top 100 universities of the World.

About VIT University

Chancellor of VIT, Dr.G.Viswanathan expressed his joy at a press conference held in New Delhi on November 16th saying that "in 1984, we began this institute as Vellore Engineering college with mere 180 students. With dedication and sincere efforts of faculties and students, we emerged as an autonomous university in the year of 2004. Apart from engineering, we offer various professional courses like catering technology, fashion designing, law and 24 different studies. In addition to this, we also provide post graduate courses like MCA, MBA, M.TECH and research courses like M.Phil and P.hd. We are proud to say that among 32000 students in our university, one third of them are female candidates. Gaining goodwill at Vellore, we thought of expanding our institution at Chennai and thus we established VIT campus at Chennai.

Following that, we recently established two more VIT campuses, one at Bhopal district in Madhya Pradesh and other at Amravati district in Andhra Pradesh. With 1682 faculties and 1100 non teaching staffs, we offer international standard education for which we have been accredited A+ by NAAC and have been recognized by ABET and IET organisations from US and UK respectively.

Last year almost 2.06 lakhs of students appeared for VIT's Engineering Entrance Exam (VITEEE) which is a national level achievement and took a prominent place in Limca Book of Records and VIT holds this position for 3 years continuously.

During the press meet Vice Presidents Mr. Sankar Viswnathan, Dr. Sekar Viswanathan, Mr. G.V. Selvam and Vice Chancellor Dr. Anand A Samuel were present.

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JEE WORKOUTS

A ER-

SECTION-

This section contains eight questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS.

1. If the roots of $10x^3 - cx^2 - 54x - 27 = 0$ are in harmonic progression, then the value of 'c' must be equal to

2. If $A = \int_0^{3\pi/2} \frac{\cos x}{\cos x - \sin x} dx$ and $B = \int_0^{3\pi/2} \frac{\sin x}{\cos x - \sin x} dx$, then the value of $A + B$ is

3. The number of integral values of k for which $x^2 - 2(4k - 1)x + 15k^2 - 2k - 7 \geq 0$ hold for all x is

4. If $\lim_{x \to 0} \frac{x^3}{\sqrt{a+x}\,(bx - \sin x)} = 1;\ a > 0$, then value of $\left[\frac{a+2b}{15}\right]$ is

5. If $k = \lim_{x \to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4};\ a > 0$ then value of $\frac{1}{\sqrt{k}} - a$ is

6. The number of complex numbers z satisfying $|z + \bar{z}| + |z - \bar{z}| = 4$ and $|z + 2i| + |z - 2i| = 4$ is/are

7. The greatest positive integer in the domain set of $f(x) = \frac{\sqrt{2+x} + \sqrt{2-x}}{x}$ is

8. If $(m^3 + 11m)x^2 + 112xy - 6(m^2 + 1)y^2 = 0$ represents a pair of perpendicular lines then sum of all possible real values of m is

SECTION-

This section contains ten questions. Each question has four options (a), (b), (c) and (d). One or more than one of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS.

9. A non-zero vector $\vec{a}$ is parallel to the line of intersection of the plane determined by the vectors $\hat{i},\ \hat{i} + \hat{j}$ and the plane determined by the vectors $\hat{i} - \hat{j},\ \hat{i} + \hat{k}$. The angle between $\vec{a}$ and the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ is

   (a) $\pi/3$ (b) $\pi/4$
   (c) $3\pi/4$ (d) none of these

10. The asymptotes of the hyperbola $xy - 3x + 4y + 2 = 0$ are

    (a) $x = -4$ (b) $x = 4$ (c) $y = -3$ (d) $y = 3$

11. Sides of a ΔABC are in A.P. If $a <$ min. $\{b, c\}$, then $\cos A =$

    (a) $\frac{4c - 3b}{2b}$ (b) $\frac{4c - 3b}{2c}$
    (c) $\frac{3c - 4b}{2c}$ (d) $\frac{4b - 3c}{2b}$

12. If A and B are two events such that $P(A \cup B) \geq \frac{3}{4}$ and $\frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$ then

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(a) $P(A)+P(B)\le\frac{11}{8}$ (b) $P(A)\cdot P(B)\le\frac{3}{8}$

(c) $P(A)+P(B)\ge\frac{7}{8}$ (d) none of these

13. If z_1, z_2, z_3 are the vertices of an equilateral triangle in the complex plane and z_0 is the centroid, then

(a) $\frac{1}{z_1-z_2}+\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}=0$

(b) $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$

(c) $z_1^2+z_2^2+z_3^2=3z_0^2$

(d) $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

14. If a, b, c are positive rational numbers such that $a > b > c$ and the quadratic equation $(a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0$ has a root in the interval $(-1, 0)$, then

(a) $b + c > a$ (b) $c + a < 2b$

(c) both roots of the given equation are rational

(d) the equation $ax^2 + 2bx + c = 0$ has both negative real roots

15. $\log_p \log_p \underbrace{\sqrt[p]{\sqrt[p]{\sqrt[p]{\cdots\sqrt[p]{p}}}}}_{n\text{ times}}$, $p>0$ and $p\ne 1$, is equal to

(a) n (b) $-n$

(c) $1/n$ (d) none of these

16. If $\frac{\ln a}{b-c}=\frac{\ln b}{c-a}=\frac{\ln c}{a-b}, a,b,c>0$, then

(a) $a^{b+c}\cdot b^{c+a}\cdot c^{a+b}=1$

(b) $a^{b+c}+b^{c+a}+c^{a+b}\ge 3$

(c) $a^{b+c}\cdot b^{c+a}\cdot c^{a+b}=3$

(d) $a^{b+c}+b^{c+a}+c^{a+b}\ge 3(3)^{1/3}$

17. $5^x+(2\sqrt{3})^{2x}-169\le 0$ is true in the interval

(a) $(-\infty, 2)$ (b) $(0, 2)$

(c) $(2, \infty)$ (d) $(0, 4)$

18. If m_1 and m_2 are the gardients of tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ which passes through $(6, 2)$ then

(a) $m_1+m_2=\frac{24}{11}$ (b) $m_1m_2=\frac{20}{11}$

(c) $m_1+m_2=\frac{48}{11}$ (d) $m_1m_2=\frac{11}{20}$

SECTION -

This section contains TWO questions. Each question contains two columns, Column I and Column II. Column I has four entries (A), (B), (C) and (D). Column II has four entries (P), (Q), (R) and (S). Match the entries in Column I with the entries in Column II. One or more entries in Column I may match with one or more entries in Column II. The ORS contains a 4 × 4 matrix whose layout will be similar to the one shown below :

For each entry in Column I, darken the bubbles of all the matching entries. For example, if entry (A) in Column I matches with entries (Q) and (R), then darken these two bubbles in the ORS. Similarly, for entries (B), (C) and (D).

(A)	(P)	(Q)	(R)	(S)
(B)	(P)	(Q)	(R)	(S)
(C)	(P)	(Q)	(R)	(S)
(D)	(P)	(Q)	(R)	(S)

19. Match the following:

	Column I		Column II
(A)	The ratio of altitude to the radius of the cylinder of maximum volume that can be inscribed in a given sphere is	P.	$\frac{1}{\sqrt{2}}$
(B)	The ratio of radius to the altitude of the cone of the greatest volume which can be inscribed in a given sphere is	Q.	$\sqrt{2}$
(C)	The cone circumscribing a sphere of radius 'r' has the minimum volume if its semi vertical angle is θ, then $30\sin\theta =$	R.	$\frac{32}{3}$
(D)	The greatest value of x^3y^4 if $2x + 3y = 7$ and $x \ge 0$, $y \ge 0$, is	S.	10

20. Match the following:

	Column I		Column II
(A)	$\lim_{x\to 0^+}\{(x\cos x)^x+(x\sin x)^{1/x}\}$ equals	P.	0
(B)	If $f(x) = x^{\sin x} + (\sin x)^{\cos x}$, then $f'(\pi/2)$ equals	Q.	1
(C)	The value of $\cot[\cot^{-1}(-1) + \cot^{-1}(-2) + \cot^{-1}(-3)]$ equals	R.	2
(D)	$\lim_{x\to\infty}\left\{\ln(2x)-\ln\left(\sqrt{x^2-1}+x\right)\right\}$ equals	S.	not defined

SECTION-

This section contains eight questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme

1. The number of ordered pairs (x, y) satisfying the system of equations $(\cos^{-1}x)^2 + \sin^{-1}y = 1$ and $\cos^{-1}x + (\sin^{-1}y)^2 = 1$ is (are)

2. If $\alpha + \beta + \gamma = \pi$ and

$$\tan\left[\frac{\alpha+\beta-\gamma}{4}\right]\tan\left[\frac{\gamma+\alpha-\beta}{4}\right]\tan\left[\frac{\gamma+\beta-\alpha}{4}\right]=1,$$

then the value of $1 + \cos\alpha + \cos\beta + \cos\gamma$ is $K - 1$, where K is

3. If the function $f(x)=\begin{cases} x+1 & \text{if} \quad x\le 1 \\ 2x+1 & \text{if} \quad 1<x\le 2 \end{cases}$

and $g(x)=\begin{cases} x^2 & \text{if } -1\le x\le 2 \\ x+2 & \text{if } 2\le x\le 3 \end{cases}$, then

the number of roots of the equation $f(g(x)) = 2$ is

4. Given that $P(3, 1)$, $Q(6, 5)$ and $R(x, y)$ are three points such that angle PRQ is a right angle and the area of $\Delta RQP = 7$, then the number of such points R is

5. Modulus of non-zero complex number 'z' satisfying $\bar{z}+z=0$ and $|z|^2-4zi=z^2$ is

6. The value of $\int\limits_{-1}^{3}(|x-2|+[x])dx$ is, (where $[\cdot]$ denotes G.I.F.)

7. If $\int (x^2+x)(x^{-8}+2x^{-9})^{1/10}dx=\frac{k}{11}(x^2+2x)+c$,

then the value of 'k' is

8. The number of solutions of the equation $\sec x+\operatorname{cosec}x=2\sqrt{2}$ in $[0, 2\pi]$ is

SECTION-

This section contains eight questions. Each question has four options (a), (b), (c) and (d). One or more than one of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS.

9. The equation $16x^2 - 3y^2 - 32x - 12y - 44 = 0$ represents a hyperbola with

(a) length of the transverse axis $2\sqrt{3}$
(b) length of the conjugate axis = 8
(c) centre at $(1, -2)$
(d) eccentricity = $\sqrt{19}$

10. If $6a^2 - 3b^2 - c^2 + 7ab - ac + 4bc = 0$ then the familiy of lines $ax + by + c = 0$, $|a| + |b| \ne 0$ is concurrent at

(a) $(-2, -3)$ (b) $(3, -1)$
(c) $(2, 3)$ (d) $(-3, 1)$

11. If the equation $\sin^{-1}(x^2 + x + 1) + \cos^{-1}(\lambda x + 1) = \pi/2$ has exactly two distinct solutions then the integral value (s) of λ can be

(a) 0 (b) 1
(c) -1 (d) 2

12. If $b > a > 0$, then the expression E given by

$$E=\frac{1}{\sqrt{b-a}}\cdot\frac{\sqrt{\frac{b-a}{a}}\sin x}{\sqrt{\left[1+\left(\sqrt{\frac{b-a}{a}}\sin x\right)^2\right]}}\sqrt{a+b\tan^2 x}$$

must be equal to

(a) $\tan x$ (b) $|\tan x|$
(c) $\frac{\sin x}{|\cos x|}$ (d) unity if $x=\frac{11\pi}{4}$

13. If A and B are two points on the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ which are farthest and nearest respectively from the point $(7, 2)$ then

(a) $A\equiv(2-2\sqrt{2},\ -3-2\sqrt{2})$
(b) $A\equiv(2+2\sqrt{2},\ -3+2\sqrt{2})$
(c) $B\equiv(2+2\sqrt{2},\ -3+2\sqrt{2})$
(d) $B\equiv(2-2\sqrt{2},\ -3-2\sqrt{2})$

14. If $\int\limits_a^b \frac{x^n}{x^n+(16-x)^n}dx=6,\ (n\in R)$ then

(a) $a = 4, b = 12$ (b) $a = 2, b = 14$
(c) $a = -4, b = 20$ (d) $a = -2, b = 18$

15. Let $f(x)$ be a continuous function for all x, which is not identically zero such that

$\{f(x)\}^2=\int\limits_0^x f(t)\cdot\frac{2\sec^2 t}{4+\tan t}dt$ and $f(0) = \ln 4$, then

(a) $f\left(\frac{\pi}{4}\right)=\ln 5$ (b) $f\left(\frac{\pi}{2}\right)=\ln 4$
(c) $f\left(\frac{3\pi}{4}\right)=\ln 3$ (d) $f(x) = 0$

16. Values of x satisfies the equation $4^{x+1.5} + 9^{x+0.5} = 10.6^x$ (is/are)

(a) $\log_{2/3}\left(\frac{3}{4}\right)$ (b) $\log_{2/3}\left(\frac{1}{2}\right)$

(c) $\log_{3/4}\left(\frac{2}{3}\right)$ (d) $\log_{1/2}\left(\frac{2}{3}\right)$

SECTION-

This section contains TWO paragraphs. Based on each paragraph, there will be TWO questions. Each question has FOUR options (a), (b), (c) and (d) out of which only one is correct. For each question, darken the bubble(s) corresponding to each the correct option in the ORS.

Paragraph for Q. No. 17 & 18

Functions in mathematics may satisfy some functional relations. For example, the function $f(x) = 3x$ satisfies $f(x + y) = f(x) + f(y)$ and from the given functional relations, we can determine several things about the functions. At times the function is completely determined if it is continuous and differentiable.

17. If $f(x + y) = f(x) + f(y)$ for all x, y, then $f\left(-\frac{3}{5}\right)$ must be equal to

(a) $f(1)$ (b) $\frac{3}{5}f(1)$

(c) $-\frac{3}{5}$ (d) $-\frac{3}{5}f(1)$

18. If $f(x)+f(y)=f\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$, then

(a) $f(4x^2 + 3x) + 3f(x) = 0$

(b) $f(3x - 4x^3) + 3f(x) = 0$

(c) $f(4x^3 + 3x) - 3f(x) = 0$

(d) $f(4x^3 - 3x) + 3f(x) = 0$

Paragraph for Q. No. 19 & 20

The limit of a function makes sense if x is defined in the neighbourhood of a. The limit of a sequence makes sense only when variable approaches ∞. A sequence $a_1, a_2, a_3, ...$ of real numbers is said to have a limit l if $\lim_{n\to\infty} a_n = l$. If l is finite the sequence is said to be convergent. If $l = \infty$ the sequence is said to be divergent. If a_n does not approach a definite number, then the sequence $\{a_n\}$ is oscillatory.

The following results are well known.

(i) $\lim_{n\to\infty} x^n = 0, |x| < 1$

(ii) If $\lim_{n\to\infty} a_n = l$, then $\lim_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n} = l$

19. Which of the following sequences does not converge to zero?

(a) $\lim_{n\to\infty} \frac{x^n}{n!}, x \in R$ (b) $\lim_{n\to\infty} \frac{\log n}{n}$

(c) $\lim_{n\to\infty} x^n, (|x|<1)$ (d) $\lim_{n\to\infty} n^{1/n}$

20. If k is an integer, such that

$$\lim_{n\to\infty}\left[\left(\cos\frac{k\pi}{4}\right)^n - \left(\cos\frac{k\pi}{6}\right)^n\right] = 0 \text{ then}$$

(a) k is divisible neither by 4 nor by 8

(b) k must be divisible by 12 but not necessarily by 24

(c) k must be divisible by 24

(d) either k is divisible by 24 or k is divisible neither by 4 nor by 6

ANS ER EY

PAPER-1

1. (9)	2. (0)	3. (3)	4. (2)
5. (6)	6. (2)	7. (2)	8. (6)
9. (b, c)	10. (a, d)	11. (b, d)	12. (a, c)
13. (a, b, c, d)	14. (b, c, d)	15. (b)	16. (a, b)
17. (a, b)	18. (a, b)		

19. (A) → Q, (B) → P, (C) → S, (D) → R

20. (A) → P, (B) → Q, (C) → P, (D) → P

PAPER-2

1. (3)	2. (1)	3. (2)	4. (0)
5. (2)	6. (7)	7. (5)	8. (3)
9. (a, b, c)	10. (a, b)	11. (a)	12. (c, d)
13. (a, c)	14. (b)	15. (a, c)	16. (a, b)
17. (d)	18. (d)	19. (d)	20. (d)

For detailed solution to the Sample Paper, visit our website, www.vidyalankar.org

MPP-7 CLASS XII ANSWER KEY

1. (c)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (b)
6. (d)	**7.** (a,c)	**8.** (c,d)	**9.** (a,b)	**10.** (a,d)
11. (a,c)	**12.** (b,c)	**13.** (a,b,c)	**14.** (b)	**15.** (d)
16. (b)	**17.** (1)	**18.** (9)	**19.** (9)	**20.** (3)

Solution Sender of Maths Musing

SET-167

1. Gouri Sankar Adhikary W.B.

SET-168

1. N. Jayanthi Hyderabad
2. Gouri Sankar Adhikary W.B.

MATHS MUSING

SOLUTION SET-168

1. (d): Let $x^2 - 10x - 45 = t$

$\therefore$ Given equation becomes

$$\frac{1}{t+16}+\frac{1}{t}-\frac{2}{t-24}=0 \Rightarrow t=-6 \Rightarrow x=13,-3$$

2. (d) : If no boy sit between the girls, then they can sit in 1, 2; 2, 3 ; . . . . ; 11, 12 = 22 ways. If one boy sits between them, they sit in 1, 3; 2, 4; . . . . . .; 10, 12 = 20 ways If two boys sit between them, they sit in 1, 4; 2, 5; .....; 9; 12 = 18 ways

The desired number is $12! - 60\cdot 10! = \frac{6}{11}\cdot 12!$

3. (c): Let $a : b : c = 1 - k : 1 : 1 + k$

$$2\sin\frac{B}{2}=\cos\frac{C-A}{2}=\frac{1}{2}\Rightarrow \sin\frac{B}{2}=\frac{1}{4}$$

$$\cos B=\frac{7}{8}=\frac{2k^2+1}{2(1-k^2)}\Rightarrow k=\frac{1}{\sqrt{5}}.$$

$$\Rightarrow\ a:b:c=\sqrt{5}-1:\sqrt{5}:\sqrt{5}+1$$

$$\therefore\ s=\frac{3\sqrt{5}}{2},\ \Delta=\frac{\sqrt{15}}{4}\Rightarrow \frac{s}{r}=\frac{s^2}{\Delta}=3\sqrt{15}.$$

4. (c): For concyclic points, $a \times 2 = (-2)(-1) \Rightarrow a = 1$.

$\therefore$ The lines are $x - 2y + 2 = 0$ and $2x - y - 3 = 0$

The circle through the four concyclic points is $(x - 2y + 2)(2x - y - 3) = 0$.

Dropping the xy term, $2(x^2 + y^2) + x + 4y - 6 = 0$

And $y = x \Rightarrow 4x^2 + 5x - 6 = 0$.

Solving, $x_1 = -2,\ x_2 = \frac{3}{4}$

If AB is the segment on $y = x$, then

$$A\equiv(-2,-2),\ B\equiv\left(\frac{3}{4},\frac{3}{4}\right)\Rightarrow AB\equiv\frac{11}{2\sqrt{2}}$$

5. (a): $Q(x) = (x^3 - 2x^2 + x - 1)(x + 1) = 0 \Rightarrow \delta = -1$

$\alpha + \beta + \gamma = 2,\ \alpha\beta + \beta\gamma + \gamma\alpha = 1$

Also, $P(x) = (x^2 + 1)\,Q(x) + (x^2 - x + 1)$

$P(\alpha) + P(\beta) + P(\gamma) + P(-1)$

$= \alpha^2 + \beta^2 + \gamma^2 + 1 - (\alpha + \beta + \gamma - 1) + 4$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) - (\alpha + \beta + \gamma) + 6 = 6$

6. (b, c): $y = 2x - x^2 \Rightarrow (x - 1)^2 = 1 - y$

$B = (1 + t, 1 - t^2),\ A = (1, 0)$

$$AB^2=t^2+(1-t^2)^2=\frac{7}{9}\Rightarrow t^2=\frac{1}{3},\frac{2}{3}$$

$$t^2=\frac{1}{3}\Rightarrow B=\left(1-\frac{1}{\sqrt{3}},\frac{2}{3}\right),\ C=\left(1+\frac{1}{\sqrt{3}},\frac{2}{3}\right)$$

$$\text{Area}=\frac{1}{2}.BC.\frac{2}{3}=\frac{2}{3\sqrt{3}}$$

$$t^2=\frac{2}{3}\Rightarrow B=\left(1-\sqrt{\frac{2}{3}},\frac{1}{3}\right),\ C=\left(1+\sqrt{\frac{2}{3}},\frac{1}{3}\right)$$

$$\text{Area}=\frac{1}{2}.BC.\frac{1}{3}=\frac{1}{3}\sqrt{\frac{2}{3}}$$

7. (a): Differentiating the given equation w.r.t. x, we get

$$2f(x)=2f'(x)+3\Rightarrow f'(x)-f(x)=-\frac{3}{2}$$

Solving the linear differential equation, we get

$$f(x)e^{-x}=-\frac{3}{2}\int e^{-x}dx+A\Rightarrow\ f(x)=\frac{3}{2}+Ae^{x}$$

$$\text{Since, } f(1)=0\Rightarrow A=-\frac{3e^{-1}}{2}$$

$$\therefore\ f(x)=\frac{3}{2}(1-e^{x-1})$$

$$a=3\int_0^x(1-e^{t-1})dt-\frac{3}{2}\int_0^1(1-e^{t-1})dt-3(1-e^{x-1})-3x=\frac{3}{2e}-3$$

8. (b) : $f'(x)=-\frac{3}{2}e^{x-1}\ \Rightarrow\ f'(1)=-\frac{3}{2}$

9. (3): $\vec{a}\cdot(\vec{a}\times\vec{r}+\vec{b})=\vec{a}\cdot\vec{r}\Rightarrow\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{r}=2$

Also, $\vec{a}\times(\vec{a}\times\vec{r}+\vec{b})=\vec{a}\times\vec{r}$

$\Rightarrow\ (\vec{a}\cdot\vec{r})\vec{a}-(\vec{a}\cdot\vec{a})\vec{r}+\vec{a}\times\vec{b}=\vec{r}-\vec{b}$

$$\therefore\ \vec{r}=\frac{1}{2}\left(2\vec{a}+\vec{b}+\vec{a}\times\vec{b}\right)$$

The projection of $\vec{r}$ along $\vec{b}$

$$=\vec{r}\cdot\frac{\vec{b}}{|\vec{b}|}=\frac{1}{2|\vec{b}|}\left(2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}\right)=\frac{4+16}{2\times4}=\frac{5}{2}$$

$\Rightarrow\ m = 5, n = 2\ \Rightarrow\ m - n = 3$

10. (b) : (P) $\binom{n}{3}\left(\frac{x}{a}\right)^{n-3}.\frac{1}{x^3}=\frac{5}{2}\Rightarrow n=6$

$$\Rightarrow\ a^3=\binom{6}{3}\frac{2}{5}=8\ \Rightarrow\ a=2$$

(Q) $(1 - x)^5\,(1 + x + x^2 + x^3)^4$

$= (1 - x)(1 - x^4)^4 = (1 - x)(1 - 4x^4 + 6x^8 - 4x^{12} + x^{16})$

$\Rightarrow$ Coeff. of x^{13} is 4

(R) $\displaystyle\sum_{k=0}^{4}\binom{4}{k}(k-2)^2=\sum_{k=0}^{4}\binom{4}{k}k^2-4\sum_{k=0}^{4}\binom{4}{k}k+4\sum_{k=0}^{4}\binom{4}{k}=16$

(S) $\displaystyle\sum_{p=1}^{4}\sum_{r=p}^{4}\binom{4}{r}\binom{r}{p}=\sum_{p=1}^{4}\sum_{r=p}^{4}\frac{4!}{p!(4-r)!(r-p)!}$

$$=\sum_{p=1}^{4}\frac{4!}{(4-p)!\,p!}\sum_{r=p}^{4}\frac{(4-p)!}{(4-r)!(r-p)!}$$

$$=\sum_{p=1}^{4}\binom{4}{p}2^{4-p}=16\sum_{p=1}^{4}\binom{4}{p}\frac{1}{2^p}=65$$